Lecture 1 (21.2.11): Simple random walk (sometimes Markov chain or general random walk)

Following Lalley: Gambler's ruin & time to hit. Stopping time & strong Markov property. ← also $\tau+\nu$, $\tau\wedge\nu$, n are stop. times.
conclude recurrence of SRW (also by summing return prob.).

First passage time dist. using generating functions). $P(\tau=2n-1)=P(S_{2n-1}=1)/(2n-1)$.
Relation to stable law of index 1/2.

Derive dist. of maximum using reflection principle. Use to give another deriv. of first passage dist. ← Catalan number & ballot thm.

Following Lalley: for general skip-free RW, use cyclic perm. trick (start at first max.) to show that $P(\tau=n)=\frac{1}{n}P(S_n=1)$. Maybe in exercise, conn. to Lagrange inv.

Wald identities: 3 identities & proofs. Use to get gambler's ruin and time again.
Possible to get first passage time dist. gen. fcn. using 3'rd identity.
Possible to get $\mathbb{E}\,S\,T\,\mathbb{1}(S_T=B)$, say, for T hitting time of $\{-A,B\}$ using third identity.

Lalley also has proof of recurrence of any mean zero 1D RW using Birkhoff's ergodic thm. and range thm. (Kesten-Spitzer-Whitman)

Also want: law of the iterated log. and no points of increase.
For iter. log., may want Azuma inequality.

Also arcsine laws for last visit to 0 and for positive time fraction.

Lecture 1 (21.2.11)

Course will focus on properties of random walk & BM.

In first half, discuss random walks focusing on SRW in $\mathbb{Z}$ and $\mathbb{Z}^d$.

In second half, define and investigate BM.

Scribe to every lecture. Possibly some HW returned at end of semester, possibly presentations by students.

Ask if know martingales.

Random Walks: $S_n = X_1 + \_\_ + X_n$ for IID X_n. $\quad \mathbb{P} = \mathbb{P}^0$, $\mathbb{E} = \mathbb{E}^0$

In most of our examples $X_1 \in \mathbb{R}^d$ or $\mathbb{Z}^d$. Sometimes more general.

Sometimes talk in generality of Markov chains on a finite or countable state space. $\mathbb{P}^x, \mathbb{E}^x$ measures when starting at x

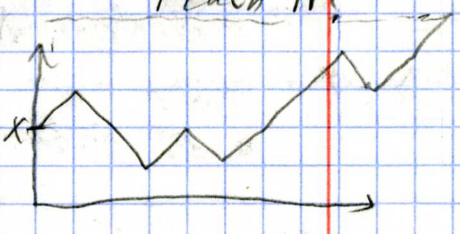
Simple random walk (SRW): $X_1 \in \{-1, 1\}$ with equal prob.

We will start by investigating basic features of SRW.

Gambler's ruin: 1) Gambler has x shekels, bet fairly one at a time until reach 0 or A. What is the chance to reach A?

More precisely: S_n SRW, $T = \min\{n \mid S_n \in \{0, A\}\}$.

Calculate $\mathbb{P}^x(S_T = A)$.



Remark: $\mathbb{P}(T < \infty) = 1$ since run of $A+1$ will occur eventually.

Define $u(x) = \mathbb{P}^x(S_T = A)$. Then $u(0) = 0$, $u(A) = 1$,
$$u(x) = \frac{1}{2}(u(x+1) + u(x-1)) \qquad 0 < x < A$$

Many ways to solve, e.g. $u(x+1) - u(x) = u(x) - u(x-1)$

$\Rightarrow u(x) = ax + b \Rightarrow u(x) = \frac{x}{A}$.

Remark: u is a harmonic function (on $\mathbb{Z}$). This is the tip of the iceberg in the relation between RW and certain PDE. Probabilistic potential theory.

2) What is the expected duration of the game, $\mathbb{E}^x T$?

Define $V(x) := \mathbb{E}^x T$, then $V(0) = 0$, $V(A) = 0$ $\quad V(x) = 1 + \frac{1}{2}(V(x+1) + V(x-1))$

$\mathbb{E}^x T = \mathbb{E}^x \mathbb{E}^x(T \mid X_1) = \mathbb{E}^x(1 + V(x + X_1)) = 1 + \frac{1}{2}(V(x+1) + V(x-1))$

or $V(x+1) + V(x-1) - 2V(x) = -2$

General sol.:

$V(x) = ax + b + f(x)$ where $f(x)$ is any fcn. such that $f(x+1)+f(x-1)-2f(x) = -2$.

Guess $f(x) = -x^2$ (operator is like second derivative).

Get $V(x) = ax + b - x^2$, $V(0) = b = 0$, $V(A) = aA + b - A^2 = 0$ $\Rightarrow$ $b=0$, $a=A$ $\Rightarrow V(x) = x(A-x)$

Exercise: Solve gambler's ruin for p-q biased walk using matrix diagonalization.

Remark: starting at $\frac{A}{2}$, $\mathbb{E}T$ is quadratic in A (maybe give as motivating question to class)

Recurrence of SRW: We obtained $P^x(\text{reach } 0) = 1$ for $x>0$ (by taking $A \to \infty$)

Translation invariance and reflection invariance implies $P^x(\text{reach } y) = 1$.

Let $\nu_y = \nu(y) = \min\{n \mid S_n = y\}$ be the first passage time to y.

It is intuitive that the walk restarts at ν_y and hence (recurrence) $P^x(\text{reach } y \text{ infinitely often}) = 1$.

For this we need:

Def.: A Stopping time for a (general) random walk (or even Markov chain) is a random variable τ taking values in $\{0,1,2,\ldots\} \cup \{\infty\}$ such that for every (finite) n, $\{\tau = n\}$ is a measurable fcn. of $S_1, \ldots, S_n$.

(in other words, if $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ then $\{\tau = n\} \in \mathcal{F}_n$
Also get $\{\tau \le n\} \in \mathcal{F}_n$, $\{\tau > n\} \in \mathcal{F}_n$)

Ex.: $\tau \wedge \nu$ is a stopping time when τ and ν are. n is a stop. time.
Hence $\tau \wedge n$ is a stop. time.

Prop. (Strong Markov property): If τ is a stopping time for $\{S_n\}_{n\ge 0}$ then on the event $\{\tau<\infty\}$ $\{S_{\tau+\cdot} - S_\tau\}$ is also a random walk with the same step dist., started at 0 and ind. of $\{S_n\}_{0\le n\le\tau}$.

More precisely, $\forall k, \ell$ $\forall x$ $s_1, \ldots, s_k, s_1^*, \ldots, s_\ell^*$ we have

$P^x(\{S_1 = s_1, \ldots, S_k = s_k\} \cap \{\tau = k\} \cap \{S_{\tau+1} - S_\tau = s_1^*, \ldots, S_{\tau+\ell} - S_\tau = s_\ell^*\}) \overset{\text{by def. of stop. time and ind. of increments}}{=}$

$P^x(\{S_1 = s_1, \ldots, S_k = s_k\} \cap \{\tau = k\}) P^0(S_1 = s_1^*, \ldots, S_\ell = s_\ell^*)$

Remark: We can define the σ-field of events known up to τ, $\mathcal{F}_\tau$, as all $A \subseteq \mathcal{F}$ such that $A \cap \{\tau \le n\} \in \mathcal{F}_n$. Then $\forall A \in \mathcal{F}_\tau$

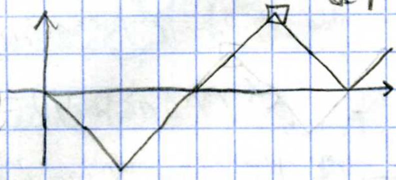
$P^x(A \cap \{\tau<\infty\} \cap \{S_{\tau+1} - S_\tau = s_1^*, \ldots, S_{\tau+\ell} - S_\tau = s_\ell^*\}) = P^x(A \cap \{\tau < \infty\}) P^0(S_1 = s_1^*, \ldots, S_\ell = s_\ell^*)$

Above recurrence is an immed. cor.

First passage time distribution for skip-free RW: A skip-free or right cont. RW is a RW with step dist X satisfying $\mathbb{P}(X \in \mathbb{Z}) = 1$, $\mathbb{P}(X \geq 2) = 0$. Let $\tau = \min\{n \mid S_n = 1\}$. We would like to find the prob. dist. of τ.

Lemma: If $x_1, \dots, x_n$ are integers ≤ 1 with sum 1 then there is a unique cyclic perm. π s.t. $\sum_{j=1}^{k} x_{\pi(j)} \leq 0 \quad \forall 1 \leq k \leq n-1$.

Proof by picture:



Start at first maximum.

exercise: complete the details.

Proposition: $\mathbb{P}(\tau = n) = \frac{1}{n} \mathbb{P}(S_n = 1)$.

proof: $\{\tau = n\} = \{\text{paths with } S_n = 1,\ S_k \leq 0\ \forall 1 \leq k \leq n-1\}$
$\{S_n = 1\} = \{\text{paths with } S_n = 1\}$.

Say two paths are equivalent if one is obtained from the other by a cyclic perm. of the increments. Note equiv. paths are equally likely.

If $p \in \{S_n = 1\}$ then all $q \sim p$ are distinct since they have a diff. pos. of first global max. Leave as exercise.

Thus $\mathbb{P}(\tau = n) = \frac{1}{n} \mathbb{P}(S_n = 1)$ as required.

Remark: This idea can be shown to be equiv to a special case of Lagrange inv.

Cor.: For SRW, $\mathbb{P}(\tau = n) = \begin{cases} 0 & n \text{ even} \\ \frac{1}{n}\binom{n}{\frac{n+1}{2}} & n \text{ odd} \end{cases}$

decays like $\frac{c}{n^{3/2}}$ on odd n. $\left(c = \frac{1}{\sqrt{\pi}}\right)$

Thus it has no expectation! (and not even $\frac{1}{2}$ moment)

Follow Lalley. Then follow Feller for ballot thm. Finally follow Lalley for Wald identities and deriv. of $\mathbb{E}s^{\tau}$. If more time, do main lemma from Feller.

Reflection principle for SRW: 1) Ballot theorem:

If candidate I gets a votes and candidate II gets b votes, $a > b$, what is the (conditioned) chance that I leads throughout the count?

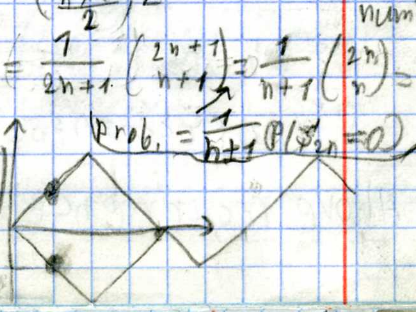
Answer: $\frac{a-b}{a+b}$. That is $\mathbb{P}(S_n = a-b, \min_{0<j<n} S_j = 1) = \frac{a-b}{a+b} \mathbb{P}(S_n = a-b)$ where $n = a+b$.

In another way $\mathbb{P}(S_n = x, \min_{0<j<n} S_j = 1) \overset{x \text{ and } n \text{ have same parity}}{=} \frac{x}{n} \binom{n}{\frac{n+x}{2}} 2^{-n}$

Corollary: $|\{\text{paths } S_{2n} = 0, S_1, \dots, S_{2n-1} \geq 0\}| = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{n+1}\binom{2n}{n} = C_n$ (Catalan number)

prob. $= \frac{1}{n+1} \mathbb{P}(S_{2n} = 0)$

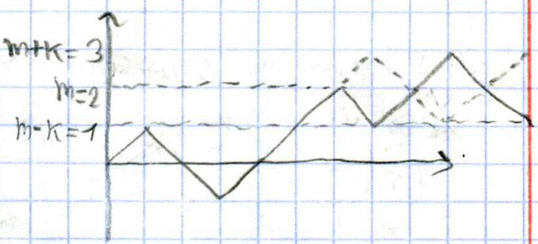
Proof: $|\{\text{paths } S_n = a-b, S_1, \dots, S_{n-1} > 0\}| \overset{\text{start at } (1,1)}{=} \binom{n-1}{a-1} - \left|\left\{\text{paths } \begin{matrix} S_0 = -1 \\ S_{n-1} = a-b \end{matrix}\right\}\right|$
$= \binom{n-1}{a-1} - \binom{n-1}{a} = \binom{n}{a} \cdot \left(\frac{a}{a+b} - \frac{b}{a+b}\right) = \binom{n}{a} \frac{a-b}{a+b}$. ($n = a+b$)



2) Maximum: Let $M_n = \max_{j \le n} S_j$.

Prop. (dist. of max. and end-point)

For $k \ge 0$, $\mathbb{P}(M_n \ge m,\ S_n = m-k) = \mathbb{P}(S_n = m+k)$



Since if we let $\tau(m) = \min(j \mid S_j = m)$

and define $S_j^* = \begin{cases} S_j & j \le \tau(m) \\ 2m - S_j & j \ge \tau(m) \end{cases}$

then S_j^* is again a SRW (André's refl. princ.) by strong Markov prop.

In particular:

Prop. (dist. of maximum): $\mathbb{P}(M_n \ge m) = \mathbb{P}(\tau(m) \le n) = \mathbb{P}(S_n = m) + 2\mathbb{P}(S_n > m) \le 2\mathbb{P}(S_n \ge m)$

Proof: $\sum_{k=-\infty}^{\infty} \mathbb{P}(M_n \ge m, S_n = m-k) = \sum_{k=0}^{\infty} \mathbb{P}(S_n = m+k) + \sum_{k=1}^{\infty} \mathbb{P}(S_n = m+k)$.

Exercise: Derive $\mathbb{P}(\tau(m) = 2n-1)$

Wald identities: Assume S_n is a general 1D RW with $S_0 = 0$.

First identity: If $\mathbb{E}|X_1| < \infty$, $\mu = \mathbb{E}X_1$, then $\forall$ stop. time τ with $\mathbb{E}\tau < \infty$ we have $\mathbb{E}S_\tau = \mu \mathbb{E}\tau$.

Second identity: If $\mathbb{E}|X_1|^2 < \infty$, $\mu = \mathbb{E}X_1$, $\sigma^2 = Var(X_1)$ then $\forall$ stop. time τ with $\mathbb{E}\tau < \infty$ we have $\mathbb{E}(S_\tau - \mu\tau)^2 = \sigma^2 \mathbb{E}\tau$.

Third identity: Assume that the mgf $\varphi(\theta) = \mathbb{E}e^{\theta X_1}$ is finite at θ, then for every bdd. stop. time τ, $\mathbb{E}\left(\frac{e^{\theta S_\tau}}{\varphi(\theta)^\tau}\right) = 1$

Remark: $\mathbb{E}\tau < \infty$ not suff. for third identity.

If $\mathbb{E}\tau = \infty$, first identity may fail. E.g. τ = hitting time of 1 for SRW

$1 = \mathbb{E}S_\tau \ne 0 = \mu \cdot \mathbb{E}\tau = 0 \cdot \infty$. In fact, for SRW and first identity, suff. that $\mathbb{E}\tau^{1/2} < \infty$.

Example: Gambler's ruin: S_n SRW. T = hitting time of $\{0, A\}$.

פרק 3 9.5

[illegible]

7.5 עד 14.5

פרק 1, [illegible]

[illegible] 25.4

[illegible] 30.4, 16.4

$\mathbb{E}^x(S_T - x) = \mu \mathbb{E}T = 0$ (easy to see $\mathbb{E}T < \infty$).

$\Rightarrow (A-x)\mathbb{P}(S_T = A) + (-x)\mathbb{P}(S_T = 0) = 0 \Rightarrow \mathbb{P}(S_T = A) = \frac{x}{A}$.

$\mathbb{E}^x(S_T - x)^2 = \sigma^2 \mathbb{E}T = \mathbb{E}T \Rightarrow \mathbb{E}T = \frac{x}{A}(A-x)^2 + \frac{A-x}{A}(-x)^2 = x(A-x)$.

Lecture 2 (28.2.2011)

Proofs of 1'st and 3'rd identity (2'nd like 1'st): (special cases of mart. thms.)

1'st identity: If τ is bdd. by M: $S_\tau = \sum_{n=0}^{M} S_n \mathbb{1}_{(\tau = n)} = \sum_{n=1}^{M} X_n \mathbb{1}_{(\tau \ge n)}$

stop. time

$\mathbb{E}S_\tau = \sum_{n=0}^{M} \mathbb{E}(X_n \mathbb{1}_{(\tau \ge n)}) = \sum_{n=1}^{M} \mathbb{E}(X_n)\mathbb{P}(\tau \ge n) = \mu \sum_{n=1}^{M} \mathbb{P}(\tau \ge n) = \mu \mathbb{E}\tau$.

Ask about make-up lecture

(since $\forall$ non-neg. RV Y, $\mathbb{E}Y = \sum_{n=1}^{\infty} \mathbb{P}(Y \ge n)$)

If τ not bdd., need to exchange $\mathbb{E}$ and sum. This is possible since

$\sum_{n=1}^{\infty} \mathbb{E}(|X_1| \mathbb{1}_{(\tau \ge n)}) = \sum_{n=1}^{\infty} \mathbb{E}|X_1| \cdot \mathbb{P}(\tau \ge n) = \mathbb{E}|X_1| \cdot \mathbb{E}\tau < \infty$.

[illegible]

114 ([illegible] 21)

204 [illegible] 13

([illegible] 30)

3'rd identity: Suppose $\tau \le M$ w/prob. 1.

$$\mathbb{E}\left(\frac{e^{\theta S_\tau}}{\varphi(\theta)^\tau}\right) = \sum_{n=0}^{M} \mathbb{E}\left(\frac{e^{\theta S_\tau}}{\varphi(\theta)^\tau} 1_{(\tau=n)}\right) = \sum_{n=0}^{M} \mathbb{E}\left(\frac{e^{\theta S_n}}{\varphi(\theta)^n} 1_{(\tau=n)}\right) =$$

$$= \sum_{n=0}^{M} \mathbb{E}\, \frac{e^{\theta S_n}}{\varphi(\theta)^n} 1_{(\tau=n)} \cdot \mathbb{E}\, \frac{e^{\theta S_M - S_n}}{\varphi(\theta)^{M-n}} \underset{\substack{\uparrow \\ \tau \text{ is a stop. time}}}{=} \sum_{n=0}^{M} \mathbb{E}\, \frac{e^{\theta S_M}}{\varphi(\theta)^M} 1_{(\tau=n)} = \mathbb{E}\, \frac{e^{\theta S_M}}{\varphi(\theta)^M} = 1.$$

Derivation of first passage time mgf using 3'rd identity

Let $\tau = \min(n \ge 0 \mid S_n = 1)$ for a SRW $\{S_n\}_{n\ge 0}$.

We would like to calculate the mgf $\mathbb{E}(s^\tau)$.

If τ was bdd., we could write $\underbrace{\mathbb{E}\left(\frac{e^{\theta S_\tau}}{\varphi(\theta)^\tau}\right) = 1}_{\text{equation for } \theta > 0}$ where $\varphi(\theta) = \mathbb{E} e^{\theta X_1}$.

First, let us see that this is sufficient to find $\mathbb{E}(s^\tau)$.

By def., $S_\tau = 1$, thus we have $\mathbb{E}\left[\left(\frac{1}{\varphi(\theta)}\right)^\tau\right] = e^{-\theta}$.

Note that $\varphi(\theta) = \frac{1}{2}(e^\theta + e^{-\theta}) = \cosh(\theta)$. Writing $s = \frac{1}{\varphi(\theta)} = \frac{2}{e^\theta + e^{-\theta}}$

we have $s(e^\theta + e^{-\theta}) = 2 \Rightarrow s(1 + e^{-2\theta}) = 2e^{-\theta} \Rightarrow e^{-2\theta} s - 2e^{-\theta} + s = 0$

$e^{-\theta} = \frac{2 \pm \sqrt{4 - 4s^2}}{2s} = \frac{1 \pm \sqrt{1 - s^2}}{s}$. The two sols. correspond to e^θ and $e^{-\theta}$.

Restricting to $\theta > 0$ we can write $e^{-\theta} = \frac{1 - \sqrt{1 - s^2}}{s}$ as it's the smaller sol. (since $s \in [0,1)$)

Thus $\mathbb{E}(s^\tau) = \frac{1 - \sqrt{1 - s^2}}{s}$ for $s \in [0,1)$ (and thus for $|s| < 1$ by analytic cont.)

It remains to show $\mathbb{E}\left(\frac{e^{\theta S_\tau}}{\varphi(\theta)^\tau}\right) = 1$ for $\theta > 0$. By the 3'rd Wald

identity $\mathbb{E}\left(\frac{e^{\theta S_{\tau\wedge n}}}{\varphi(\theta)^{\tau\wedge n}}\right) = 1 \quad \forall n \ge 0$. Since $\mathbb{P}(\tau < \infty) = 1$,

$\lim_{n\to\infty} \frac{e^{\theta S_{\tau\wedge n}}}{\varphi(\theta)^{\tau\wedge n}} = \frac{e^{\theta S_\tau}}{\varphi(\theta)^\tau}$ a.s. Noting that $0 < \frac{1}{\varphi(\theta)} \le 1$ and

$0 < e^{\theta S_{\tau\wedge n}} \le e^\theta$, the equality follows by the dominated conv. thm.

Sample path prop. of SRW

Lemma: $\mathbb{P}(S_1 \ne 0, \dots, S_{2n} \ne 0) = \mathbb{P}(S_{2n} = 0)$

Proof: Enough to show

$\mathbb{P}(S_1 > 0, \dots, S_{2n} > 0) = \frac{1}{2}\mathbb{P}(S_{2n} = 0)$

Already saw in ballot theorem: $\mathbb{P}(S_1 > 0, \dots, S_{2n-1} > 0, S_{2n} = 2r) \overset{(r \ge 1)}{=} \frac{r}{n}\mathbb{P}(S_{2n} = 2r) =$

$= \frac{1}{2}\left(\mathbb{P}(S_{2n-1} = 2r-1) - \mathbb{P}(S_{2n-1} = 2r+1)\right)$

part of ballot thm. proof

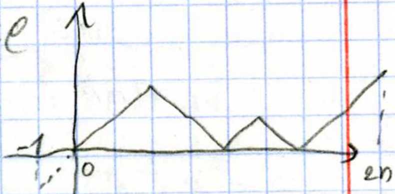
Thus: $P(S_1>0,\dots,S_{2n}>0)=\sum_{r=1}^{\infty}\frac{1}{2}\left(P(S_{2n-1}=2r-1)-P(S_{2n-1}=2r+1)\right)=\frac{1}{2}P(S_{2n-1}=1)$

and we are done since $P(S_{2n}=0)=\frac{1}{2}\left(P(S_{2n-1}=1)+P(S_{2n-1}=-1)\right)=P(S_{2n-1}=1)$.

Note also that $P(S_1\geq 0,\dots,S_{2n}\geq 0)=P(S_{2n}=0)$ since

$\frac{1}{2}P(S_{2n}=0)=P(S_1>0,\dots,S_{2n}>0)=\frac{1}{2}P(S_1\geq 0,\dots,S_{2n-1}\geq 0)=$

Parity $\rightarrow$ $=\frac{1}{2}P(S_1\geq 0,\dots,S_{2n}\geq 0)$



This gives another version of the first passage time dist.

Let $T=\min(n\geq 1 \mid S_n=0)$ then $P(T=2n)=P(S_1\neq 0,\dots,S_{2n-2}\neq 0)-P(S_1\neq 0,\dots,S_{2n}\neq 0)$

$=P(S_{2n-2}=0)-P(S_{2n}=0)=\frac{1}{2n-1}P(S_{2n}=0)$.

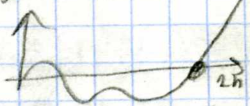
Arcsine laws - last zero and time above the axis

Consider last visit to zero up to time $2n$. Surprisingly, dist. is sym. around n and achieves max. at edges.

For example (Feller): If tossing a coin every second for a year,

$P(\text{last } 0 \text{ before } 9 \text{ days})\approx\frac{1}{10}$, $P(\text{before } 2\frac{1}{4} \text{ days})\approx\frac{1}{20}$,

$P(\text{before } 2 \text{ hours and } 10 \text{ minutes})\approx\frac{1}{100}$.



(arcsine dist. for last zero)

Thm: Let $T=\max(k\leq n \mid S_{2k}=0)$ (T is not a stop. time!)

then $P(T=2k)=P(S_{2k}=0)P(S_{2n-2k}=0)$.

Remark: $P(S_{2k}=0)=\binom{2k}{k}2^{-2k}\sim\frac{1}{\sqrt{\pi k}}$ as $k\to\infty$.

thus $P(T=2k)\approx\frac{1}{\pi\sqrt{k(n-k)}}=\frac{1}{n}\cdot\frac{1}{\pi\sqrt{\frac{k}{n}(1-\frac{k}{n})}}$. Note that

$$\int_0^x\frac{1}{\pi\sqrt{y(1-y)}}\,dy=\frac{2}{\pi}\arcsin(\sqrt{x})$$

Corollary: $\lim_{\substack{n\to\infty\\ k/n\to x}}P(T\leq 2k)=\frac{2}{\pi}\arcsin(\sqrt{x})$.

Markov prop.

Proof of thm: $P(T=2k)=P(S_{2k}=0,S_{2k+1}\neq 0,\dots,S_{2n}\neq 0)=$

$=P(S_{2k}=0)\cdot P(S_1\neq 0,\dots,S_{2n-2k}\neq 0)\underset{\text{lemma}}{=}P(S_{2k}=0)\cdot P(S_{2n-2k}=0)$

Thm. (arcsine dist. for time above axis): $=|\{0\leq m\leq 2n-1 \mid S_m\geq 0, S_{m+1}\geq 0\}|$

$P(\text{time spent above axis}=2k)=P(S_{2k}=0)\cdot P(S_{2n-2k}=0)$.

Similar Corollary

Example (Feller): as before, if 2 players play every second for a year there is chance $\frac{1}{10}$ that the less fortunate player will lead for less than 9 days.

Denote $\mathbb{P}(\text{Time above axis} = 2k) =: b_{2k,2n}$

Proof of thm.: For $k=n$ $\quad b_{2n,2n} = \mathbb{P}(S_1 \geq 0, \ldots, S_{2n} \geq 0) = \mathbb{P}(S_{2n} = 0)$

same for $k=0$. Now take $0 < k < n-1$. Denote also

$f_{2r} = \mathbb{P}(\text{first return to } 0 \text{ at time } 2r)$. Then

$$b_{2k,2n} = \mathbb{P}(\text{Time above axis} = 2k) = \sum_{r=1}^{n} \mathbb{P}(\text{time} = 2k, \text{first return} = 2r) =$$

$$= \sum_{r=1}^{n} f_{2r}\, \mathbb{P}(\text{time} = 2k \mid \text{first return} = 2r) \overset{\text{symmetry}}{=} \frac{1}{2}\sum_{r=1}^{k} f_{2r} b_{2k-2r,2n-2r} + \frac{1}{2}\sum_{r=1}^{n-k} f_{2r} b_{2k,2n-2r}$$

Assume by induction that for all $1 \leq \nu \leq n-1$ we have

shown $b_{2k,2\nu} = \mathbb{P}(S_{2k} = 0)\mathbb{P}(S_{2\nu-2k} = 0)$. Then

$$b_{2k,2n} = \frac{1}{2}\sum_{r=1}^{k} f_{2r}\,(\mathbb{P}(S_{2k-2r} = 0)\mathbb{P}(S_{2n-2k} = 0)) + \frac{1}{2}\sum_{r=1}^{n-k} f_{2r}\mathbb{P}(S_{2k} = 0)\mathbb{P}(S_{2n-2k-2r} = 0)$$

$$= \frac{1}{2}\mathbb{P}(S_{2n-2k} = 0)\sum_{r=1}^{k} f_{2r}\mathbb{P}(S_{2k-2r} = 0) + \frac{1}{2}\mathbb{P}(S_{2k} = 0)\sum_{r=1}^{n-k} f_{2r}\mathbb{P}(S_{2n-2k-2r} = 0) =$$

$$= \frac{1}{2}\mathbb{P}(S_{2n-2k} = 0)\,\mathbb{P}(S_{2k} = 0) + \frac{1}{2}\mathbb{P}(S_{2k} = 0)\mathbb{P}(S_{2n-2k} = 0)$$

Time reversal

There are 2 more arc-sine laws obtained from the first two using time reversal.

Denoting the reversed path by $\{S_k^*\}$ we have $S_k^* = S_n - S_{n-k}$.

Thus, for example, $\{S_j^* > 0 \ \forall 1 \leq j \leq n\} = \{S_n > S_j \ \forall 0 \leq j \leq n-1\}$ and

replacing n by $2n$ we have $\mathbb{P}(S_{2n} > S_j \ \forall 0 \leq j \leq 2n-1) = \mathbb{P}(S_j^* > 0 \ \forall 1 \leq j \leq 2n) = \frac{1}{2}\mathbb{P}(S_{2n} = 0)$

or ladder point record at time $2n$ (and a similar formula for records at odd times)

weak ladder point

Also $\mathbb{P}(S_{2n} \geq S_j \ \forall j) = \mathbb{P}(S_{2n} = 0)$ similarly

arc-sine law for first visit to terminal point:

$$\mathbb{P}(\min(j \mid S_{2j} = S_{2n}) = k) = \mathbb{P}(S_{2k} = 0)\mathbb{P}(S_{2n-2k} = 0)$$

dual to dist. of last zero.

arc-sine law for position of first/last global maxima

For simplicity, take an even length walk and suppose the first maximum occurs at $2k$. Then we have:

$$\underbrace{S_0 < S_{2k}, \ldots, S_{2k-1} < S_{2k}}_{\text{Prob.} = \frac{1}{2}\mathbb{P}(S_{2k} = 0)}, \underbrace{S_{2k+1} \leq S_{2k}, \ldots, S_{2n} \leq S_{2k}}_{\text{Prob.} = \mathbb{P}(S_{2n-2k} = 0)}$$

Thus $\mathbb{P}(\text{first max. at } 2k) = \frac{1}{2}\mathbb{P}(S_{2k} = 0)\mathbb{P}(S_{2n-2k} = 0)$

The same is true for last max. at $2k$ (moving the $\frac{1}{2}$ between the factors).

A similar statement holds for max. at an odd location.

Random walk bridge equidistribution of positive time

In striking contrast to the arc-sine law for ordinary SRW, for a SRW bridge we have (an analogous thm. holds also for the pos. of a uniformly chosen global maximum by considering cyclic perm.)

Thm.: $\mathbb{P}(\text{time spent above axis} = 2k \mid S_{2n}=0) = \frac{1}{n+1}$.

Let $g_{2k,2n} = \mathbb{P}(\text{time above axis} = 2k \mid S_{2n}=0) = \frac{\mathbb{P}(\text{time above axis} = 2k, S_{2n}=0)}{\mathbb{P}(S_{2n}=0)}$ $\leftarrow d_{2k,2n}$

For $k=n$, by ballot thm. corollary, $d_{2n,2n} = \frac{1}{2n+1}\binom{2n+1}{n+1}2^{-2n} = \underbrace{\frac{1}{n+1}\binom{2n}{n}2^{-2n}}_{g_n} = \frac{1}{n+1}\mathbb{P}(S_{2n}=0)$

Similarly for $k=0$. Let $1 \le k < n$, $f_{2r} = \mathbb{P}(\text{first return to } 0 \text{ is at } 2r)$.

$$d_{2k,2n} = \sum_{r=1}^{n} \mathbb{P}(\text{time}=2k, S_{2n}=0, \text{first return}=2r) = \frac{1}{2}\sum_{r=1}^{k} f_{2r} d_{2k-2r,2n-2r} + \frac{1}{2}\sum_{r=1}^{n-k} f_{2r} d_{2k,2n-2r}$$

as in arc-sine law proof

By induction: $d_{2k,2n} = \frac{1}{2}\sum_{r=1}^{k} f_{2r}\cdot\frac{\mathbb{P}(S_{2n-2r}=0)}{n-r+1} + \frac{1}{2}\sum_{r=1}^{n-k} f_{2r}\frac{\mathbb{P}(S_{2n-2r}=0)}{n-r+1}$

Recalling that $f_{2r} = \frac{1}{2}\cdot\frac{1}{r}\mathbb{P}(S_{2r-2}=0)$ $[= \frac{1}{2r-1}\mathbb{P}(S_{2r}=0)]$ by ballot thm. cor. again, we have

$$d_{2k,2n} = \frac{1}{4}\sum_{r=1}^{k}\frac{1}{r(n-r+1)}\mathbb{P}(S_{2r-2}=0)\mathbb{P}(S_{2n-2r}=0) + \frac{1}{4}\sum_{r=1}^{n-k}\frac{\mathbb{P}(S_{2r-2}=0)\mathbb{P}(S_{2n-2r}=0)}{r(n-r+1)} =$$

replace r by $n-r+1$ in second sum $\rightarrow$ $= \frac{1}{4}\sum_{r=1}^{n}\frac{\mathbb{P}(S_{2r-2}=0)\mathbb{P}(S_{2n-2r}=0)}{r(n-r+1)}$ ind. of k!

Lecture 3 (7.3.11)

Reminder: no class next week. Pass contact page (name, email, registered, may register). Find a scribe.

Mention that results of E. Sparre Andersen give a combinatorial approach to proving some of the arcsine laws for general $\mathbb{R}$-RW even without moment assumptions.

Law of the iterated logarithm

For a SRW (S_n), by the CLT, we have S_n is approx. $N(0,n)$.

Since if $N \sim N(0,n)$ we have $\mathbb{P}(N \in (k-\frac{1}{2}, k+\frac{1}{2})) \approx \frac{1}{\sqrt{2\pi n}}e^{-k^2/2n}$

we may expect the same for S_n. Note however that S_n necessarily has the parity of n. Thus, we may expect $\mathbb{P}(S_n=k) \approx \begin{cases} 0 & k \text{ of diff. parity from } n \\ \sqrt{\frac{2}{\pi n}}e^{-k^2/2n} & \text{o/w} \end{cases}$

Similarly, $\mathbb{P}(N>k) \underset{k/\sqrt{n}\to\infty}{\sim} \frac{n}{k}\cdot\frac{1}{\sqrt{2\pi n}}e^{-k^2/2n}$ and we may expect something similar from S_n.

(LCLT) **Lemma:** i) For $k=o(n^{3/4})$ of the same parity as n, $\mathbb{P}(S_n=k) \sim \sqrt{\frac{2}{\pi n}}e^{-k^2/2n}$ $\left(\substack{n\to\infty \\ k=o(n^{3/4})}\right)$

ii) For all $n, k \ge 0$ $\mathbb{P}(S_n \ge k) \le e^{-k^2/2n}$.

Proof: i) $\mathbb{P}(S_n=k) = \binom{n}{(n+k)/2}2^{-n} = \frac{n!\,2^{-n}}{\left(\frac{n+k}{2}\right)!\left(\frac{n-k}{2}\right)!} \underset{n\to\infty,\, k=o(n^{3/4})}{\sim} \frac{\sqrt{2\pi n}}{\sqrt{2\pi\frac{n+k}{2}\,2\pi\frac{n-k}{2}}}\cdot\frac{\left(\frac{n}{e}\right)^n 2^{-n}}{\left(\frac{n+k}{2e}\right)^{(n+k)/2}\left(\frac{n-k}{2e}\right)^{n-k/2}} = \sqrt{\frac{2}{\pi n(1-\frac{k^2}{n^2})}}\frac{n^n}{(n+k)^{n+k/2}\cdot(n-k)^{n-k/2}}$.

K and n have same parity

$\exp(n\log n - \frac{n+k}{2}\log(n+k) - \frac{n-k}{2}\log(n-k))$

$n\log n - (\frac{n+k}{2})\log(n+k) - (\frac{n-k}{2})\log(n-k) = -\frac{n+k}{2}\log(1+\frac{k}{n}) - \frac{n-k}{2}\log(1-\frac{k}{n}) =$

$\log(1-x) = -(x+\frac{x^2}{2}+\frac{x^3}{3}+O(x^4))$ as $x\to 0$

$= -\frac{n+k}{2}\left(\frac{k}{n}-\frac{k^2}{2n^2}+\frac{k^3}{3n^3}+O(\frac{k^4}{n^4})\right) + \frac{n-k}{2}\left(\frac{k}{n}+\frac{k^2}{2n^2}+\frac{k^3}{3n^3}+O(\frac{k^4}{n^4})\right) =$ (as $\frac{k}{n}\to 0$)

$= -\frac{k^2}{2n}+O(\frac{k^4}{n^3})$

Thus $P(S_n=k) \underset{\substack{n\to\infty\\ k=o(n^{3/4})}}{\sim} \sqrt{\frac{2}{\pi n}}\, e^{-\frac{k^2}{2n}+O(\frac{k^4}{n^3})} \sim \sqrt{\frac{2}{\pi n}}\, e^{-\frac{k^2}{2n}}$

ii) special case of Bernstein-Chernoff-Hoeffding-Azuma inequality.

Note $\frac{1}{2}e^{\theta}+\frac{1}{2}e^{-\theta} \le e^{\theta^2/2}$ by comparing Taylor coef. $\left(\frac{\theta^{2n}}{(2n)!}\le\frac{\theta^{2n}}{2^n n!}\right)$

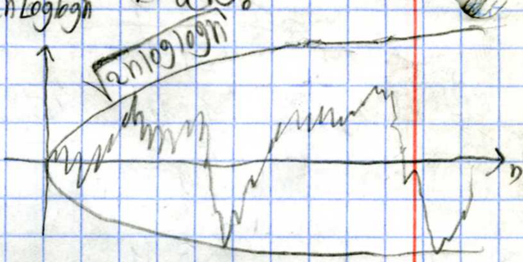
$P(S_n\ge k) \underset{\theta\ge 0}{\le} \frac{\mathbb{E}e^{\theta S_n}}{e^{\theta k}} = \frac{(\frac{1}{2}e^{\theta}+\frac{1}{2}e^{-\theta})^n}{e^{\theta k}} \le e^{\theta^2 n/2-\theta k} \underset{\theta=\frac{k}{n}\ (k\ge 0)}{=} e^{-k^2/2n}$

Reminder: $M_n := \max_{0\le j\le n} S_j$, $P(M_n\ge k) = P(S_n=k)+2P(S_n>k)$

Exercise: $P(A_n \text{ occur inf. often}) \ge \limsup P(A_n)$. Gives lower bound of $c\sqrt{n}$ in LIL.

Theorem (Law of the iter. log.) $\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}} = 1$ a.s.

Remark: By symmetry $\liminf_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}} = -1$ a.s.



(We will ignore issues of integrality of n in S_n in the proof)

Idea: $P(S_{a^k}\ge c\sqrt{2a^k\log\log(a^k)}) \approx e^{-\frac{c^2a^k\log\log(a^k)}{a^k}} = \left(\frac{1}{k\log a}\right)^{c^2}$ summable iff $c>1$.

These are relatively ind. events and the walk does not fluctuate much between the a^k. This already gives upper bound on subsequence.

Lemma: For any $0<\epsilon<1$, $\limsup_{n\to\infty}\max_{n\le j\le(1+\epsilon)n}\frac{|S_j-S_n|}{\sqrt{2n\log\log n}} \le 4\sqrt{\epsilon}$ a.s.

Proof: Fix $\epsilon,\alpha>0$, $n_k=(1+\epsilon)^k$, then

$P\left(\max_{n_k\le j\le n_{k+1}}|S_j-S_{n_k}|\ge\alpha\sqrt{2n_k\log\log(n_k)}\right) \le 4P\left(S_{n_{k+1}-n_k}\ge\alpha\sqrt{2n_k\log\log(n_k)}\right) \le$

$\le 4e^{-\frac{\alpha^2 n_k\log\log(n_k)}{n_{k+1}-n_k}} = 4e^{-\frac{\alpha^2\log\log(n_k)}{\epsilon}} = 4\left(\frac{1}{k\log(1+\epsilon)}\right)^{\frac{\alpha^2}{\epsilon}}$

Use better proof below!!

So if $\alpha^2>\epsilon$, a.s. only finitely many of these events occur by the Borel-Cantelli lemma. Thus for all large k

$\max_{n_k\le j\le n_{k+1}}|S_j-S_{n_k}|\le\alpha u(n_k)$ $(\alpha^2>\epsilon)$ where $u(s)=\sqrt{2s\log\log s}$

Noting that for any $q>1$ and k large enough: $\frac{u(q^{k+1})}{u(q^k)} = \frac{u(q^{k+1})}{q^{k+1}}\cdot\frac{q^k}{u(q^k)}q \le q$

Since $\frac{u(t)}{t}$ is decreasing in t for t large enough.

we see that if $n_k\le n\le n_{k+1}$ and $n\le j\le(1+\epsilon)n$ and k is large:

$|S_j-S_n|\le|S_j-S_{n_{k+1}}|+|S_{n_{k+1}}-S_{n_k}|+|S_{n_k}-S_n|\le 2\alpha u(n_k)+\alpha u(n_{k+1})\le$

$\leq (2\alpha + \alpha(1+\varepsilon))u(n_k) \leq 4\alpha u(n_k)$

Taking $\alpha \downarrow \sqrt{\varepsilon}$ over a sequence proves the lemma.

Proof of thm.: For any $\gamma > 0$, $\limsup_{a>1} S_{a^k} \leq (1+\gamma)u(a^k)$ by the proof idea above and Borel-Cantelli. Taking $a = 1+\varepsilon$ we have by the lemma for $a^k \leq n \leq a^{k+1}$: $\frac{S_n}{u(n)} = \frac{S_{a^k}}{u(a^k)} \frac{u(a^k)}{u(n)} + \frac{S_n - S_{a^k}}{u(a^k)} \cdot \frac{u(a^k)}{u(n)} \leq 1 + \gamma + 4\sqrt{\varepsilon}$

Taking $\varepsilon, \gamma \to 0$ over a subsequence proves the upper bound in the thm.

For the lower bound note that $P(S_{a^k} - S_{a^{k-1}} \geq (1-\gamma)u(a^k - a^{k-1})) \geq$ for $0<\gamma<1$, $a>1$ (large)

$\geq \sqrt{\frac{2}{\pi(a^k - a^{k-1})}} e^{-(1-\gamma)^2 \log\log(a^k - a^{k-1})}$ which is not summable in k. Hence by the B-C lemma, a.s. for large k, $S_{a^k} - S_{a^{k-1}} \geq (1-\gamma)u(a^k - a^{k-1})$.

By the upper bound for $\varepsilon > 0$ a.s. for large k, $S_{a^k} \geq -(1+\varepsilon)u(a^k)$. Thus

$$\frac{S_{a^k}}{u(a^k)} \geq (1-\gamma)\frac{u(a^k - a^{k-1})}{u(a^k)} + \frac{S_{a^{k-1}}}{u(a^k)} \geq (1-\gamma)\frac{u(a^k - a^{k-1})}{u(a^k)} - (1+\varepsilon)\frac{u(a^{k-1})}{u(a^k)} \xrightarrow[k\to\infty]{} (1-\gamma)\sqrt{1-\frac{1}{a}} - \frac{1+\varepsilon}{\sqrt{a}}$$

Thus $\limsup_{n\to\infty} \frac{S_n}{u(n)} \geq (1-\gamma)\sqrt{1-\frac{1}{a}} - \frac{1+\varepsilon}{\sqrt{a}}$. Taking $\gamma \to 0$, $a \to \infty$ over a subsequence proves the lower bound in the thm.

Better proof for upper bound: $\gamma>0, a>1$: $P(\max_{0\leq n\leq a^k} S_n \geq (1+\gamma)u(a^k)) \leq 2e^{-\frac{(1+\gamma)^2 u(a^k)^2}{2a^k}} =$

$= 2\left(\frac{1}{k\log a}\right)^{(1+\gamma)^2}$ is summable, so occurs only finitely many times a.s.

For large n, write $a^{k-1} \leq n \leq a^k$ and note

$$\frac{S_n}{u(n)} = \frac{S_n}{u(a^k)} \cdot \frac{u(a^k)}{a^k} \cdot \frac{a^k}{n} \cdot \frac{n}{u(n)} \leq (1+\gamma) \cdot a$$ since $\frac{u(n)}{n} \geq \frac{u(a^k)}{a^k}$ for large n.

Thus $\limsup_{n\to\infty} \frac{S_n}{u(n)} \leq (1+\gamma)a$ a.s. Taking $\gamma \to 0$, $a \to 0$ over a sequence, the upper bound is proved.

Higher dimensions and more general RW (Durrett chap 4)

Recurrence/transience

A RW taking values in $\mathbb{R}^d$ is called (point) recurrent if $P(S_n = 0 \text{ i.o.}) = 1$. Otherwise transient.

We also define the set $P := \{x \in \mathbb{Z}^d \mid \exists n \text{ s.t. } P(S_n = x) > 0\}$, the set of possible values for S_n. A state x is called (point-) recurrent if $P(S_n = x \text{ i.o.}) = 1$

Put discussion after next page.

The strong Markov property implies that the walk is recurrent iff all its possible states are recurrent.

EX.: If one state is recurrent then all possible states are recurrent.

Can it be that $0 < P(S_n = 0 \text{ i.o.}) < 1$? The next thm. shows it is impossible.

say that an event is <u>permutable</u> if its occurrence for a walk with increments $(X_1, X_2, \dots)$ is unaffected by applying a finite perm. to the increments. In other words, if our increments take values in S (in our case, $\mathbb{Z}^d$ or $\mathbb{R}^d$) and we take $\Omega = S^{\mathbb{N}}$ with the prob. measure coming from our increments then $A \in \mathcal{F}$ is perm. if $\pi A = \{\omega \mid \pi^{-1}\omega \in A\} = A$ for all finite perm. π.

The collection of permutable events forms the exchangeable σ-field $\mathcal{E} \subseteq \mathcal{F}$. For example: $A = \{S_n = 0 \text{ i.o.}\}$, $A = \{\limsup \frac{S_n}{c_n} \geq 1\}$.

All tail events are permutable so tail σ-field $\subseteq \mathcal{E}$.

<u>Thm. (Hewitt-Savage 0-1 law)</u>: For a RW (or an IID sequence), if $A \in \mathcal{E}$ then $\mathbb{P}(A) \in \{0,1\}$.

<u>Proof</u>: Fix $A \in \mathcal{E}$. We will show $\mathbb{P}(A) = \mathbb{P}(A)^2$. Let $A_n \in \sigma(X_1, \dots, X_n)$

(1) be such that $\mathbb{P}(A_n \Delta A) \to 0$ (where $A_n \Delta A = (A \setminus B) \cup (B \setminus A)$ so that $1_{A_n \Delta A} = |1_{A_n} - 1_A|$)
$= (A \cup B) \setminus (A \cap B)$

Let $\pi = \pi_n$ be the perm. which exchanges coords. $1..n$ with $n+1..2n$.

Write $A_n' = \pi A_n$. Since π preserves measure

(2) $\mathbb{P}(A_n \Delta A) = \mathbb{P}(\pi(A_n \Delta A)) = \mathbb{P}(A_n' \Delta A)$ permutability of A.

Since $|\mathbb{P}(B) - \mathbb{P}(C)| \leq \mathbb{P}(B \Delta C)$ (triangle inequality in $1_{B\Delta C} = |1_B - 1_C|$)

(3) (1) and (2) imply $\mathbb{P}(A_n), \mathbb{P}(A_n') \to \mathbb{P}(A)$. However, it also implies
$\mathbb{P}(A_n \Delta A_n') \leq \mathbb{P}(A_n \Delta A) + \mathbb{P}(A_n' \Delta A) \to 0$. From which
$0 \leq \mathbb{P}(A_n) - \mathbb{P}(A_n \cap A_n') \leq \mathbb{P}(A_n \cup A_n') - \mathbb{P}(A_n \cap A_n') = \mathbb{P}(A_n \Delta A_n') \to 0$
implying $\mathbb{P}(A_n \cap A_n') \xrightarrow{(3)} \mathbb{P}(A)$. Recalling $\mathbb{P}(A_n \cap A_n') = \mathbb{P}(A_n)\mathbb{P}(A_n')$ by ind., we are done by (3).

<u>Application</u>: For a RW in $\mathbb{R}$, exactly one of the following has prob. 1:
i) $S_n = 0 \; \forall n$. ii) $S_n \to \infty$ iii) $S_n \to -\infty$ iv) $\liminf S_n = -\infty$, $\limsup S_n = \infty$.

<u>Proof</u>: $\limsup S$ is a constant $c \in [-\infty, \infty]$ by the prev. thm.
$c \in \{-\infty, \infty\}$ unless $X_1 \equiv 0$ since $c = c - X_1$. The same applies to $\liminf S_n$, giving the result.

Ex.: Show that if $X_1 \in \mathbb{R}$ is non-degenerate and either i) symmetric ii) has $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 < \infty$ then we are in case (iv) (actually $\mathbb{E}X_1 = 0$ suffices as we will see)

Lecture 4 21.3.11

Scribe Continuing from recurrence discussion two pages ago.

Fix mistake from last time in LIL

Prop.: A RW in $\mathbb{R}^d$ is point-recurrent iff one of the following holds:

i) $P(\exists n \geq 1\ S_n = 0) = 1$ ii) $P(S_n = 0 \text{ i.o.}) = 1$ iii) $\sum_{n=0}^{\infty} P(S_n = 0) = \infty$.

Proof: By the Strong Markov Property the number of returns to the origin is dist. $Geom(p)$ with $p = P(\text{no return to } 0)$.

(iii) follows since $\mathbb{E}\binom{\text{number of}}{\text{returns}} = \sum_{n=0}^{\infty} P(S_n = 0) = \frac{1}{p}$.

Remark: If the RW has a cont. dist., a more useful notion is neighbourhood-recurrence: $\forall \varepsilon > 0\ P(|S_n| < \varepsilon \text{ i.o.}) = 1$.

One can similary define possible values using neighbourhoods and similar prop. are valid.

Durrett chap. 4

Polya's thm ($\sim$1920): SRW on $\mathbb{Z}^d$ is recurrent iff $d = 1, 2$. (drunk man/bird Durrett cites Kakutani)

Proof: In 1D we already saw this but we can get this also by Stirling since $P(S_n = 0) \sim \frac{1}{\sqrt{\pi n}}$. For $d=2$, note that by rotating the lattice by $45°$, the walk is equiv to making one step each turn in each of 2 ind. SRW.

Hence $P(S_{2n} = 0) = \left(\frac{1}{\sqrt{\pi n}}\right)^2$. For $d=3$:

$$P(S_{2n} = 0) = 6^{-2n} \sum_{\substack{j,k \geq 0 \\ j+k \leq n}} \frac{(2n)!}{(j!\,k!\,(n-j-k)!)^2} = 2^{-2n}\binom{2n}{n} \sum_{j,k} \Big(\underbrace{3^{-n} \frac{n!}{j!\,k!\,(n-j-k)!}}_{\text{sum to } 1}\Big)^2 \leq$$

$$\leq 2^{-2n}\binom{2n}{n} \max_{j,k} 3^{-n} \frac{n!}{j!\,k!\,(n-j-k)!}$$

It remains to show $\max_{j,k} \text{---} \leq \frac{C}{n}$ for some $C > 0$. Indeed, the max. occurs when the 3 numbers are as close as possible to $\frac{n}{3}$ and the result follows from Stirling.

For $d \geq 4$, observe the walk is transient since we can ignore the last $d-3$ coord. and it will still be transient.

Chung-Fuchs theorem ($\sim$1950): A RW in $\mathbb{R}^d$ is neighbourhood-recurrent (point-recurrent if on $\mathbb{Z}^d$ follows as cor.) if $d = 1$ and $\frac{S_n}{n} \to 0$ in Prob. (weak law of large numbers) or if $d = 2$ and $\frac{S_n}{\sqrt{n}} \Rightarrow$ normal dist.

If $d \geq 3$ and the random walk is not contained in a plane then it is (neighbourhood-) transient.

Exercise: show in 1D that it does not imply point-rec. for RW in $\mathbb{Z}$.

SLLN, KSW thm. and proof of first part on $\mathbb{Z}^d$ when mean $=0$. Stable law counterexample

Shepp (1964): $\exists$ recurrent RW in 1D with arbitrary large tails.

Strong law of large numbers: If $\mathbb{E}X_1 = \mu$ (in 1D) then $\frac{S_n}{n} \to \mu$ a.s.

Implies that if $\mu \neq 0$ then $S_n \to \infty$ or $S_n \to -\infty$ a.s. according to sign of μ.
In particular, no recurrence.

Birkhoff ergodic thm. (for factor of IID): Given $X_1, X_2, \ldots$ IID RVs (say in $\mathbb{R}$) and $g: \mathbb{R}^\infty \to \mathbb{R}$ measurable.

If we define $Y_n = g(X_n, X_{n+1}, \ldots)$ and if $\mathbb{E}|Y_1| < \infty$

then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n Y_k = \mathbb{E}Y_1$ a.s. and in L_1.

Remark: sufficient to have the Y's be any stat. ergodic seq. with $\mathbb{E}|Y_1| < \infty$. We will not prove the thm. in our course.

Application: Theorem (Kesten-Spitzer-Whitman) (1964)

For a RW S_n on $\mathbb{R}^d$, let R_n be the number of distinct sites visited by time n, $R_n = |\{S_0, \ldots, S_n\}|$

then $\frac{R_n}{n} \to \mathbb{P}(\text{no return to zero})$ a.s. (and in L_1 by bdd. conv.

Remark: True for any Markov chain or even stat. seq.

Proof: $R_n = \sum_{j=0}^n 1\{S_j \text{ not revisited by time } n\}$.

$R_n \geq \sum_{j=0}^n 1\{S_j \text{ never revisited}\}$ and by the ergodic thm.:

$$\liminf_{n\to\infty} \frac{R_n}{n} \geq \mathbb{P}(S_0 \text{ never revisited}) = \mathbb{P}(\text{no return to } 0)$$

For the other direction, fix $M \geq 1$.

$$R_n \leq \sum_{j=0}^{n-M} 1(S_j \text{ not revisited by time } j+M) + M$$

by the ergodic thm.:

$$\limsup_{n\to\infty} \frac{R_n}{n} \leq \mathbb{P}(0 \text{ not revisited by time } M+1) \searrow_{M\to\infty} \mathbb{P}(\text{no return to } 0)$$

since event on right is intersection of events on left.

Proof of recurrence thm. for $d=1$ when mean $=0$:

If X_1 has mean 0 then $\frac{S_n}{n} \to 0$ a.s. by the SLLN.

Hence for any $\varepsilon > 0$, $\limsup_{n\to\infty} \frac{R_n}{n} \leq \varepsilon$. Thus $\frac{R_n}{n} \to 0 \overset{KSW}{=} \mathbb{P}(\text{no return to } 0)$.

Example that SLLN not equiv. to WLLN: X_1 sym. with $\mathbb{P}(X_1>x)\underset{x\to\infty}{\sim}\frac{1}{x\log x}$

Then weak law holds but strong law does not. Can prove by Fourier analysis?

However, weak law criterion is not sharp for recurrence.

For cont. RW (and neighbourhood recurrence), can consider X_1 symmetric stable of index α. That is $\mathbb{E}e^{itX_1}=e^{-|t|^\alpha}$.

As is well-known $\mathbb{E}|X_1|^p<\infty$ $\forall p<\alpha$, $\mathbb{E}|X_1|^\alpha=\infty$ and the weak law holds iff $\alpha>1$ (since it is equivalent to the Fourier transform diff. at 0, and the deriv. is the value that $\frac{S_n}{n}$ is close to - theorem of E. Pitman).

For $\alpha=1$, this is the Cauchy dist., with density $\frac{1}{\pi(1+x^2)}$.

From the Fourier trans. criterion we will mention, these walks are neighbourhood-recurrent iff $\alpha\geq 1$ (with $\alpha=1$ included)

Stone (Shepp 64): $\exists$ 1D RW with arbitrary large tails which are point-recurrent. That is, $\forall\varepsilon(x)\underset{x\to\infty}{\searrow}0$ $\exists X_1$ s.t. $\mathbb{P}(X_1>x)\geq\varepsilon(x)$ for large x, and S_n point-recurrent.

Give as exercise with hint.

Correction from last class: We showed for SRW, if $k=o(n^{3/4})$ has the same parity as n then $\mathbb{P}(S_n=k)\sim\sqrt{\frac{2}{\pi n}}e^{-k^2/2n}$ $n\to\infty$.

In fact, we had this estimate uniformly in k for $k=o(n^{3/4})$.

Lemma: For $k=o(n^{3/4})$, $\mathbb{P}(S_n\geq k)\geq c\frac{\sqrt{n}}{k}e^{-k^2/2n}$ if $k>\sqrt{n}$.

Proof: $\mathbb{P}(S_n\geq k)\geq\mathbb{P}(k\leq S_n\leq k+(\frac{n}{k}\wedge\sqrt{n}))\geq\frac{c}{\sqrt{n}}\sum_{\substack{k\leq j\leq k+(\frac{n}{k}\wedge\sqrt{n})\\ j\text{ of same parity as }n}}e^{-j^2/2n}$

Noting that for these j, if $k\leq\sqrt{n}$, we have

$$e^{-j^2/2n}\geq e^{-(k+\sqrt{n})^2/2n}=e^{-\frac{k^2}{2n}-\frac{k}{\sqrt{n}}-\frac{1}{2}}\geq ce^{-k^2/2n}$$

similarly, if $k>\sqrt{n}$, $e^{-j^2/2n}\geq e^{-(k+\frac{n}{k})^2/2n}=e^{-\frac{k^2}{2n}-1-\frac{n}{2k^2}}\geq ce^{-\frac{k^2}{2n}}$.

Thus $\mathbb{P}(S_n\geq k)\geq\frac{c}{\sqrt{n}}e^{-k^2/2n}\cdot(\frac{n}{k}\wedge\sqrt{n})\geq\begin{cases}c & \text{if } k<\sqrt{n}\text{ (also follows from CLT)}\\ c\frac{\sqrt{n}}{k}e^{-k^2/2n} & \text{if } k>\sqrt{n}\end{cases}$

In proof of LIL, we had $A_k = \{ S_{a^k} - S_{a^{k-1}} \geq (1-\delta) u(a^k - a^{k-1}) \}$ for $u(t) = \sqrt{2t \log\log t}$, $0<\delta<1$ and $a>1$, and we needed to show $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty$. Indeed, by prev. lemma for large k:

$$\mathbb{P}(A_k) \geq c \frac{\sqrt{a^k - a^{k-1}}}{(1-\delta) u(a^k - a^{k-1})} e^{-(1-\delta)^2 u^2 (a^k - a^{k-1}) / 2(a^k - a^{k-1})} =$$

$$= C' \frac{1}{(1-\delta)\sqrt{\log\log(a^k - a^{k-1})}} \frac{1}{[\log(a^k - a^{k-1})]^{(1-\delta)^2}}$$

which is not summable in k.

Proof of recurrence thm. in $d=2$:

Let $G(x,y) = \mathbb{E}^x \#\text{visits to } y = \sum_{n=0}^{\infty} \mathbb{P}^x(S_n = y)$.

Lemma: $G(x,y) = \mathbb{P}^x(\text{visit } y) \cdot G(x,x) \leq G(x,x)$.

Proof: Let $T = \min\{n / S_n = y\}$. Then

$$G(x,y) = \sum_{n=0}^{\infty} \mathbb{P}^x(S_n = y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{P}^x(S_n = y, T=k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{P}^x(S_n = y, T=k) \overset{\text{strong Markov property}}{=}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{P}^x(T=k) \cdot \mathbb{P}^y(S_n = y) = \sum_{k=0}^{\infty} \mathbb{P}^x(T=k) \underbrace{G(y,y)}_{=G(x,x)} = \mathbb{P}^x(\text{visit } y) G(x,x).$$

(Durrett Chap. 4) Proof of Thm.: Need to show that $\sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) = G(0,0) = \infty$.

Have $\forall b>0$, $\mathbb{P}\left(\frac{|S_n|}{\sqrt{n}} \leq b\right) \to \int_{|x|\leq b} n(x)\,dx$ where n is the normal density.

Note that if we had an LCLT of form $\mathbb{P}(S_n = 0) \geq \frac{c}{n}$, we would be done. The idea is to get something similar from the CLT.

By the prev. lemma: $\sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) \geq \frac{c}{m^2} \sum_{n=0}^{\infty} \mathbb{P}(|S_n| \leq m)$

(if we had, $\mathbb{P}(S_n = 0) \geq \frac{1}{n} \mathbb{P}(|S_n| \leq \sqrt{n})$, it would also suffice).

Note $\frac{1}{m^2} \sum_{n=0}^{\infty} \mathbb{P}(|S_n| \leq m) = \int_0^{\infty} \mathbb{P}(|S_{\lfloor \theta m^2 \rfloor}| \leq m)\,d\theta$. By Fatou: $\liminf \frac{1}{m^2} \sum_{n=0}^{\infty} \mathbb{P}(|S_n| \leq m) \geq$

$$\left(\mathbb{P}(|S_{\lfloor \theta m^2 \rfloor}| \leq m) = \mathbb{P}\left(\frac{|S_{\lfloor \theta m^2 \rfloor}|}{\sqrt{\lfloor \theta m^2 \rfloor}} \leq \frac{m}{\sqrt{\lfloor \theta m^2 \rfloor}}\right), \quad \frac{m}{\sqrt{\lfloor \theta m^2 \rfloor}} \to = \frac{1}{\theta^{1/2}} \cdot \frac{\lfloor \theta m^2 \rfloor}{\theta m^2} \right)$$

$$\geq \int_0^{\infty} \int_{|x| \leq \theta^{-1/2}} n(x)\,dx\,d\theta \underset{\exists\, \zeta, c>0}{\geq} \int_{\zeta}^{\infty} \frac{c}{\theta} n(0)\,d\theta = \infty. \text{ Thus } \sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) = \infty.$$

Fourier analytic criterion for recurrence: (for RW on $\mathbb{Z}^d$)

Thm: Let $\varphi(\theta) = \mathbb{E} e^{i\theta \cdot X_1}$ be the char. fcn. of the RW.

Then S_n is recurrent iff $\lim_{r \uparrow 1} \int_{[-\pi,\pi]^d} \mathrm{Re}\left[\frac{1}{1 - r\varphi(\theta)}\right] d\theta = \infty$

Remark: 1) A much more difficult thm. is that the prev. cond. can be replaced by $\int_{[-\pi,\pi]^d} \mathrm{Re}\left[\frac{1}{1-\varphi(\theta)}\right] d\theta = \infty$.

2) A similar statement is true for RW on $\mathbb{R}^d$. There

the walk is neighbourhood-recurrent iff for some $\delta>0$ (then for all δ) we have $\int_{(-\delta,\delta)^d} \frac{1}{1-\varphi(\theta)} d\theta = \infty$ (or the stateme with r

<u>Proof:</u> recall that $\varphi^n(\theta) = \mathbb{E}e^{i\theta \cdot S_n}$ and by the Inv. formula

$$\mathbb{P}(S_n = y) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} e^{-iy\cdot\theta} \varphi^n(\theta) d\theta.$$

Thus $\mathbb{P}(S_n = 0) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \varphi^n(\theta) d\theta$ we have

$$\sum_{n=0}^{\infty} r^n \mathbb{P}(S_n = 0) = \sum_{n=0}^{\infty} (2\pi)^{-d} \int_{[-\pi,\pi]^d} r^n \varphi^n(\theta) d\theta \underset{\text{abs. summable}}{=} (2\pi)^{-d} \int_{[-\pi,\pi]^d} \frac{1}{1-r\varphi(\theta)} d\theta$$

Since the LHS is real

$$\sum_{n=0}^{\infty} r^n \mathbb{P}(S_n = 0) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \mathrm{Re}\left[\frac{1}{1-r\varphi(\theta)}\right] d\theta$$

and we conclude since $\lim_{r\nearrow 1} \sum_{n=0}^{\infty} r^n \mathbb{P}(S_n=0) = \sum_{n=0}^{\infty} \mathbb{P}(S_n = 0)$

<u>Lecture 5</u> 28.3.11

Scribe: Give proof from prev. page of 2D Chung-Fuchs thm.
Remind Fourier analytic recurrence criterion.
Perhaps mention that in 2D, criterion applied to sym. stable dist. with char. fcns. $\varphi(t) = e^{-|t|^\alpha}$ $t\in\mathbb{R}^2$, $|t| = \sqrt{t_1^2+t_2^2}$ $(\alpha \le 2)$
is neighbourhood-recurrent only in the Gaussian case $\alpha = 2$.

<u>Def.</u>: A RW in 3D is <u>truly 3-D</u> if $\mathbb{P}(X_1 \cdot t \neq 0) > 0$ $\forall t \neq 0$.
(Does not stay in a plane)

<u>Proof of 3D Chung-Fuchs thm.</u>: Idea: in 1D If $\varphi(t) = \mathbb{E}e^{itX_1} = 1 + o(t^2)$ $(t\to 0)$

(based on Durrett Chap. 4) then $X_1 \equiv 0$ since it implies first and second moments are 0.

In more than 1D, same applies in each direction $\varphi(t\theta) = \mathbb{E}e^{it(\theta\cdot X_1)}$ ← Char. fcn. of $\theta\cdot X_1$.

1) $\mathrm{Re}\frac{1}{1-z} \le \frac{1}{\mathrm{Re}(1-z)}$ if $\mathrm{Re}(z)\le 1$. both ∞ if $z=1$, only RHS ∞ if $z\neq 1$, $\mathrm{Re}(z)=1$. <u>Proof</u>: Write $1-z = re^{i\theta}$ with $\theta\in[-\pi/2,\pi/2]$.
Then $\frac{1}{1-z} = \frac{1}{r}e^{-i\theta}$, $\mathrm{Re}(\frac{1}{1-z}) = \frac{1}{r}\cos\theta$, $\mathrm{Re}(1-z) = r\cos\theta$, $\frac{1}{\mathrm{Re}(1-z)} = \frac{1}{r\cos\theta}$ Assume $|\theta|\neq\frac{\pi}{2}$ by comment
Need $\cos\theta \le \frac{1}{\cos\theta}$. True if $\cos\theta > 0$.

2) $1-\cos(x) \ge x^2/4$ for $|x|\le\frac{\pi}{3}$. <u>Proof</u>: $1-\cos x = \int_0^x \sin y\, dy = \int_0^x\int_0^y \cos z\, dz \underset{|x|\le\pi/3}{\ge} \int_0^x\int_0^y \frac{1}{2} dz = \frac{x^2}{4}$.

3) $\underset{\mathrm{Re}(1-\varphi(t))}{\mathrm{Re}(1-\mathbb{E}e^{it\cdot X_1})} = \mathbb{E}(1-\cos(t\cdot X_1)) \ge \mathbb{E}\left(1_{|t\cdot X_1|\le\frac{\pi}{3}} \cdot \frac{(t\cdot X_1)^2}{4}\right)$

Write $t = ru$ with $r>0$, $u \in S^2$ and observe

$$\mathrm{Re}(1-\varphi(ru)) \geq \frac{r^2}{4} \mathbb{E}\left(1_{\{r|u\cdot X_1| \leq \frac{\pi}{3}\}} \cdot (u \cdot X_1)^2\right)$$

4) (By continuity of $\mathbb{E}(u\cdot X_1)^2$ in u and compactness of S^2
$\inf_{u\in S^2} \mathbb{E}(u\cdot X_1)^2 = \min_{u \in S^2} \mathbb{E}(u\cdot X_1)^2 > 0$ since X_1 is truly 3-D.) not needed

Furthermore, since if $r\to 0$ and $u(r)\to u$ we have by Fatou

$$\liminf_{r\to 0} \mathbb{E}\left(1_{\{r|u\cdot X_1| \leq \frac{\pi}{3}\}} (u\cdot X_1)^2\right) \geq \mathbb{E}(u\cdot X_1)^2 > 0 \text{ since } X_1 \text{ is truly 3-D.}$$

Hence by compactness $\exists r_0$ s.t. $\forall r<r_0$, $u\in S^2$, $\mathbb{E}(1_{\{r|u\cdot X_1| \leq \frac{\pi}{3}\}}(u\cdot X_1)^2) > c > 0$

5) Thus (3) and (4) give $\mathrm{Re}(1-\varphi(ru)) \geq c r^2$ for $r<r_0$.

6) It follows that $\int_{[\pi,\pi]^3} \mathrm{Re}\left(\frac{1}{1-\rho\varphi(t)}\right) dt \underset{\rho<1}{\leq} \int_{[\pi,\pi]^3} \frac{1}{\mathrm{Re}(1-\varphi(t))} dt \geq \int_{\substack{t\in[\pi,\pi]^3 \\ |t|<r_0}} \frac{1}{C|t|^2} dt =$

$= \int_0^{r_0} \frac{r^2}{cr^2} dr < \infty$, implying the walk is transient.

(Note that in d-dim., result follows from 3D result, but also get $\int_0^{r_0} \frac{r^{d-1}}{cr^2} dr$ in the calculation for a truly d-dim. walk.

martingales (need conv.)
harmonic fcns. and tail σ-field. and coupling Remark
Trivial on rec. conn. graphs. (in Durrett 6.7) (Lawler 1.4)
harmonic fcns. off a set in $\mathbb{Z}^d$.
Kaimanovich-Vershik → Beginning potential theory and Martin kernel paper.
Sublinear harmonic fcns., non-neg. harmonic fcns.
Liouville
Poisson boundary

Review of martingales in discrete time

Def: Filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$ σ-fields. equality mod 0.

Martingale: $\{M_n\}_{n\geq 0}$ seq. of integrable RV's with M_n (cond. exp. only def. mod 0.)
wrt. filtration $\{\mathcal{F}_n\}_{n\geq 0}$ meas. wrt. $\mathcal{F}_n$ and $\forall m\leq n$ $M_m = \mathbb{E}(M_n|\mathcal{F}_m)$

Supermart.: $\forall m\leq n$ $M_m \geq \mathbb{E}(M_n|\mathcal{F}_m)$, Submart.: $\forall m \leq n$ $M_m \leq \mathbb{E}(M_n|\mathcal{F}_m)$

Sufficient to check: $\forall n$ $M_n = \mathbb{E}(M_{n+1}|\mathcal{F}_n)$

If the filtration is not specified then $\mathcal{F}_n = \sigma(M_0, \ldots, M_n)$.

If $M_0, X_1, X_2, \ldots$ ind., $\mathbb{E}|M_0|<\infty$ and $\mathbb{E}X_i = 0$ $\forall i$ then
$M_n := M_0 + X_1 + \ldots + X_n$ is a mart.

Prop: Jensen's ineq. gives that if $f:\mathbb{R}\to\mathbb{R}$ is convex then if $\{M_n\}_{n\geq 0}$ is a mart. and $\mathbb{E}|f(M_n)|<\infty$ $\forall n$ then $\{f(M_n)\}$ is a submart.

In particular, can take $\{|M_n|^\alpha\}_n$ for $\alpha\ge 1$ and $\{e^{bM_n}\}_n$ for $b\in\mathbb{R}$ and get a submart. (assuming these are int.)

Optional sampling

Def.: Stopping time T wrt. the filtration $\{\mathcal{F}_n\}_{n\ge 0}$ is a $\{0,1,\dots\}\cup\{\infty\}$ valued RV s.t. $\forall n$, $\{T\le n\}$ is $\mathcal{F}_n$-meas.

Note that the min. of two stop. times is a stop. time and that const. are stop. time.

Prop.: $\{M_n\}_{n\ge 0}$ mart., T stop. time $\Rightarrow \{M_{T\wedge n}\}_{n\ge 0}$ is a mart. — same for submartingale or superm.

In part., $\mathbb{E}(M_{T_n}) = \mathbb{E}M_0$.

Proof: Immediate from def. $M_{T\wedge(n+1)} = M_{T\wedge n}\mathbb{1}_{(T\le n)} + M_{n+1}\mathbb{1}_{(T\ge n+1)}$

Thm. (optional sampling thm.): $\{M_n\}_{n\ge 0}$ mart. and T stop. time wrt. $\{\mathcal{F}_n\}_{n\ge 0}$. Assume $\mathbb{P}(T<\infty)=1$ and at least one of the following holds:

1) T is bdd. (i.e., $\mathbb{P}(T\le K)=1$ for some K).

2) (Dom. conv.) $\exists Y\in L_1$ s.t. $|M_{T_n}|\le Y$.

3) $\mathbb{E}|M_T|<\infty$ and $\lim_{n\to\infty}\mathbb{E}(|M_n|\mathbb{1}_{(T>n)})=0$.

4) $\{M_n\}$ are unif. int. I.e., $\forall\varepsilon>0\ \exists K_\varepsilon$ s.t. $\mathbb{E}(|M_n|\mathbb{1}_{(M_n>K_\varepsilon)})<\varepsilon\ \forall n$.

5) $\exists\alpha>1$ s.t. $\{M_n\}_{n\ge 0}$ are bdd. in L_α ($\mathbb{E}|M_n|^\alpha\le K\ \forall n$).

6) $\mathbb{E}T<\infty$ and $\{M_n\}$ has bdd. diff. ($|M_n-M_{n-1}|\le K$ $\forall n\ge 1$ and some K.)

Then $\mathbb{E}M_0=\mathbb{E}M_T$ (in part., $M_T\in L_1$). For submart., $\mathbb{E}M_0\le\mathbb{E}M_T$.

Proof: 1, 2 clear. $5\Rightarrow 4$. For 3, $M_T = M_{T\wedge n} + M_T\mathbb{1}_{(T>n)} - M_n\mathbb{1}_{(T>n)}$.

Since $\mathbb{1}_{(T>n)}\to 0$ a.s. and $\mathbb{E}|M_T|<\infty$, dom. conv. gives $\lim_{n\to\infty}\mathbb{E}M_T\mathbb{1}_{(T>n)}=0$. Hence 3 is suff. 4 (A14 in Williams). $4\Rightarrow 3$? (Lawler-Limic claim so)

6: $|M_{T\wedge n}-M_0| = |\sum_{k=1}^{T\wedge n}(M_k-M_{k-1})|\le KT$. Dom. conv. gives $\mathbb{E}(M_T-M_0)=0$

Doob's maximal inequality: If $\{M_n\}_{n\ge 0}$ is a nonneg. submart. wrt. $\{\mathcal{F}_n\}$ and $\lambda>0$ then $\mathbb{P}(\max_{0\le j\le n}M_j\ge\lambda)\le\frac{\mathbb{E}M_n}{\lambda}$.

Especially useful as: If $\{M_n\}$ is a mart., $\alpha\ge 1$, $b\ge 0$: $\mathbb{P}(\max_{0\le j\le n}|M_j|\ge\lambda)\le\frac{\mathbb{E}|M_n|^\alpha}{\lambda^\alpha}$, $\mathbb{P}(\max_{0\le j\le n}M_j\ge\lambda)\le\frac{\mathbb{E}e^{bM_n}}{e^{b\lambda}}$.

Martingale conv. thm.: 1) $\{M_n\}_{n\ge 0}$ supermart., $\sup_n\mathbb{E}(M_n^-)<\infty$ (e.g. $M_n\ge 0$ $\forall n$) then ($M_n^- = \max(-M_n,0)$) $\exists$ RV M s.t. $M_n\to M$ a.s. Convergence need not occur in L_1! (e.g. SRW stopped when hitting 1.) However, $M\in L_1$ (Durrett 5.2.8)

2) For a submart., TFAE: 1) UI 2) conv. a.s. and in L_1 3) conv. in L_1.

For a mart., also: 4) $\exists M\in L_1$, $M_n=\mathbb{E}(M\mid\mathcal{F}_n)$.

Williams: $\triangleright$ signifies something important $\triangleright\triangleright$ something very important $\triangleright\triangleright\triangleright$ the mart. conv. thm.

Cor.: If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ (in part., $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$) then $\forall X \in L_1$,

$\mathbb{E}(X|\mathcal{F}_n) \xrightarrow[n\to\infty]{} \mathbb{E}(X|\mathcal{F}_\infty)$ a.s. and in L_1.

Case $X = 1_A$ called Lévy's 0-1 law. Generalizes Kolmogorov 0-1 law.

Harmonic functions

A function $h: G \to \mathbb{R}$ on a graph G (loc. fin. (always assumed)) is called harmonic if $\forall x$

$h(x) = \frac{1}{|N(x)|} \sum_{y \in N(x)} h(y)$ where $N(x) = \{y \mid y \sim x\}$. In other words, if $\{Z_n\}_{n\geq 0}$ is a SRW on G, for any starting vertex x, $h(x) = \mathbb{E}^x h(Z_1)$

Thus if $h(Z_n)$ is int. $\forall n$, then $\{h(Z_n)\}_{n\geq 0}$ is a mart..

Examples on $\mathbb{Z}^d$: constants, linear fcns.

Def. The space of all bdd harmonic fcns. on G is called the Poisson boundary of G. G is called Liouville if the only bdd. harmonic fcns. on it are constants.

Remark. Usually studied for the Cayley graphs of groups. Sometimes harmonicity is defined wrt. other RW's.

Prop. On any conn. recurrent graph, any non-neg. harmonic fcn. is constant.

Got to here but also showed example of bdd. harmonic fcn. on tree

Proof: Fix $x, y \in G$ and a non-neg. harmonic fcn. h. Let $M_n := h(Z_n)$ with $Z_0 \equiv x$. (M_n is bdd. and hence int. since G is loc. fin.) By the mart. conv. thm., $M_n \to M$ a.s..

Since G is conn. recurrent, $\mathbb{P}(Z_n = x \text{ i.o.}) = \mathbb{P}(Z_n = y \text{ i.o.}) = 1$.

Hence $\mathbb{P}(M = h(x)) = \mathbb{P}(M = h(y)) = 1$. Thus $h(x) = h(y)$.

Def.: A coupling of two dist. μ, ν is a joint dist. (X, Y) with $X \sim \mu$ and $Y \sim \nu$.

Aldous-Thorisson (93) paper, Lindvall (92), Thorisson book, Kaim. Vershik paper (83)

Lecture 6 4.4.11 (scribe) (HW)

Shift-coupling with better in context of Markov chains. Same as two times (T, T') s.t. $Z^x_T = Z^y_{T'}$ and $\mathbb{P}(T<\infty) = \mathbb{P}(T'<\infty) = 1$

Prop: If for any $x, y \in G$ $\exists$ coupling of the SRW Z^x_n, Z^y_n with $Z^x_0 \equiv x$, $Z^y_0 = y$ s.t. $\mathbb{P}(\exists K$ for which $Z^x_n = Z^y_{n+K}$ for inf. many $n) = 1$ then any bdd. harmonic fcn. on G is constant.

Proof: $M^x_n := h(Z^x_n)$, $M^y_n := h(Z^y_n)$ are mart. conv. to M^x, M^y resp.

By assumption, under our coupling $\mathbb{P}(M^x_n = M^y_{n+K}$ for inf. many n and some $K) = 1$.

Hence $\mathbb{P}(M^x = M^y) = 1$. Since M^x_n and M^y_n are bdd. and hence UI, we get $h(x) = \mathbb{E}M^x = \mathbb{E}M^y = h(y)$.

Prop.: On $\mathbb{Z}^d$ there are no bdd. harmonic fcns.

Proof: It is sufficient to find a shift-coupling for any two starting points $x, y \in \mathbb{Z}^d$. First assume $x-y$ has all even coord. Do the following coupling, at each step Z_n^x and Z_n^y move in the same coord., ind. if that coord. is diff. in Z_n^x and Z_n^y and together otherw. Since 1D SRW is recurrent, this gives a shifted coupling (with no shift). If not all coords. of $x-y$ are even, run Z_n^x until time K when $Z_n^x - y$ has all coords. even, then do prev. coupling. Since $P(K<\infty)=1$, this gives a shifted coupling with shift K.

Harmonic fcns, σ-fields and coupling

We describe the study of harmonic fcns. in the wider context of Markov chains, though our main example will still be RW on graphs (with discrete topology). Let $(E, \mathcal{E})$ be a countable state space, $p: S \times S \to [0,1]$ a Markov kernel ($\sum_y p(x,y) = 1 \ \forall x$). The space of trajectories is S^∞ with the product σ-algebra $\mathcal{E}^\infty$. **Def:** $\theta: S^\infty \to S^\infty$, $\theta((x_n)_{n\geq 0}) = (x_{n+1})_{n\geq 0}$, the shift.

$U \in \mathcal{T}$ RV, a fcn. of any tail of the trajectory → **Tail σ-field:** $\mathcal{T} = \bigcap_{n=0}^{\infty} \theta_n^{-1}(\mathcal{E}^\infty)$. Inside the intersection is the σ-field of events depending only on coords. from n onward.

$U \in \mathcal{I}$ RV, a fcn. of the traj. inv. under shifts. **Invariant σ-field:** $\mathcal{I} = \{B \in \mathcal{E}^\infty \mid \theta_1^{-1} B = B\}$, all events holding for a trajectory iff they hold for any shift of it.

Note: $\mathcal{I} \subseteq \mathcal{T}$ as can be easily verified.

Examples: 1) SRW on $\mathbb{Z}^d$: the parity of the starting point X_0 can be deduced from observing X_n for any n, hence it is in $\mathcal{T}$. Hence if the initial dist. ν is supp. on both even and odd points then this event has non trivial prob. under P^ν.

2) In a periodic MC, the initial period is in the tail σ-field.

3) SRW on a complete binary tree: The walk eventually fixates on some end. The event that it finishes in the bottom half or any end

is invariant under shifts and hence belongs to $\mathcal{I}$. It has non-trivial prob. under any P^x. Here too, parity is in the tail σ-field.

Def.: Identification mod 0: We shall identify two RV's $U, V \in \mathcal{E}^\infty$ if for any $x \in E$ $P^x(U \neq V) = 0$. We may write $U = V$ (mod 0).

The following is the fundamental relation between these σ-fields and bounded harmonic fcns.

Def: A P-harmonic fcn. $h: E \to \mathbb{R}$ is a fcn. satisfying $h(x) = \sum_{y \in E} h(y) P(x,y)$ $\forall x \in E$.

A P-spacetime-harmonic fcn. $h: E \times \{0,1,\ldots\} \to \mathbb{R}$ is a fcn. satisfying

$h(x,n) = \sum_{y \in E} h(y, n+1) P(x,y)$ $\forall x \in E, n \geq 0$. ← Example: parity of Z_n, say.

Thm: 1) There is a bijection between bounded $U \in \mathcal{T}$ considered mod 0 and bounded spacetime harmonic fcns.

given by $U \in \mathcal{T} \Rightarrow h(x,n) = \mathbb{E}^x(U_n)$ where $U = U_n \circ \theta^n$.

2) There is a bijection between bounded $U \in \mathcal{I}$ mod 0 and bdd. harmonic fcns., $U \in \mathcal{I} \Rightarrow h(x) = \mathbb{E}^x U$.

Proof 1) ⇒: $\underset{\text{bounded}}{U \in \mathcal{T}}$. Define $h(x,n) = \mathbb{E}^x(U_n)$ where $U = U_n \circ \theta^n$.

h is clearly bdd.. $h(x,n) = \mathbb{E}^x(U_n) = \mathbb{E}^x \mathbb{E}^x(U_n | Z_1) \underset{U \in \mathcal{T}}{\overset{\uparrow}{=}} \mathbb{E}^x \mathbb{E}^{Z_1}(U_{n+1}) = \mathbb{E}^x h(Z_1, n+1)$

So that h is spacetime harmonic.

⇐: $h(x,n)$ bdd. spacetime harmonic. For each x, $(h(Z_n^x, n))_{n \geq 0}$ is a bdd. martingale and hence converges. The limit is U under $\mathbb{P}^x$.

By def. as a limit, it is a tail RV.

Finally, since h is bdd., we also have $h(Z_n^x, n) = \mathbb{E}^x(U | \mathcal{F}_n)$.

So that U is indeed the limit of h and the map $U \to h$ is 1-1.

2) Similar proof. $h(x) = \mathbb{E}^x U = \mathbb{E}^x \mathbb{E}^x(U | Z_1) \underset{U \in \mathcal{I}}{\overset{\uparrow}{=}} \mathbb{E}^x \mathbb{E}^{Z_1} U = \mathbb{E}^x h(Z_1)$.

Then give thm. with many equiv. and then examp. Then Kaimanovich-Vershik (ask Gidi too). Finally Lawler for Dirichlet problem.

Couplings: Prop.: If $\forall x, y \in E$ $\exists$ coupling of Z_n^x, Z_n^y with $T := \min\{n \mid Z_n^x = Z_n^y\}$ satisfying $\mathbb{P}(T < \infty) = 1$ (a successful coupling) then all bdd. spacetime harmonic fcns are constant.

Proof: Fix x, y. $h(Z_n^x, n)$ and $h(Z_n^y, n)$ are bdd. martingales.

$|h(x,0) - h(y,0)| = |\mathbb{E} h(Z_n^x, n) - \mathbb{E} h(Z_n^y, n)| \leq |\mathbb{E}(h(Z_n^x, n) - h(Z_n^y, n)) 1_{(T > n)}| \leq$

$\leq 2M\mathbb{P}(T > n)$ where $M = \max_{z,n} h(z,n)$. Hence $h(x,0) = h(y,0)$ $\forall x, y$.

Similarly using that $h(Z_n^x, n+m)$ is a mart. for every m gives $h(x,m) = h(y,m)$ $\forall x, y, m$.

Prop: If $\forall x, y \in E$ $\exists$ coupling of Z_n^x, Z_n^y s.t. $\mathbb{P}(\exists k \text{ s.t. } Z_{n+k}^x = Z_n^y \text{ for all large enough } n) = 1$ (a successful shift-coupling) then all bdd. harmonic fcns. are constant.

proof: Similar to prev. case.

In fact, these statements are iff and we have many other equivalences:

Thm.: The following are equivalent for a Markov chain

1) Trivial tail σ-field for every initial distribution.
2) All bdd. spacetime harmonic fcns. are constant.
3) $\exists$ a successful coupling of Z_n^x, Z_n^y $\forall x,y$.
4) The Markov chain is mixing.
5) $\|\mathbb{P}^{\nu}(Z_n \in \cdot) - \mathbb{P}^{\nu'}(Z_n \in \cdot)\|_{TV}$ $\|\nu P^n - \nu' P^n\|_{TV} \xrightarrow[n\to\infty]{} 0$ $\forall$ initial ν, ν'
6) $\|\mathbb{P}^{\nu}(\theta^n Z \in \cdot) - \mathbb{P}^{\nu'}(\theta^n Z \in \cdot)\|_{TV} \xrightarrow[n\to\infty]{} 0$ $\forall \nu, \nu'$ measures on processes.

1) Trivial invariant σ-field for any initial dist.
2) All bdd. harmonic fcns. are constant.
3) $\exists$ a successful shift-coupling of Z_n^x, Z_n^y $\forall x,y$.
4) The Markov chain is Cesaro-mixing.
5) $\frac{1}{t}\|\int_0^t \nu P^s ds - \int_0^t \nu' P^s ds\|_{TV} \xrightarrow[t\to\infty]{} 0$ $\forall \nu, \nu'$
6) $\|\mathbb{P}^{\nu}(\theta^{\lfloor Ut \rfloor} Z \in \cdot) - \mathbb{P}^{\nu'}(\theta^{\lfloor Ut \rfloor} Z \in \cdot)\|_{TV} \xrightarrow[t\to\infty]{} 0$ where $U \sim \text{Unif}(0,1)$ is ind. of Z.

proof of (5) from (3): (3) gives a coupling for any initial dist.

Allows to prove rate of conv. results!

$|\mathbb{P}^{\nu}(Z_n \in A) - \mathbb{P}^{\nu'}(Z_n \in A)| = |\mathbb{E}((1_{(Z_n^{\nu} \in A)} - 1_{(Z_n^{\nu'} \in A)}) 1_{(T > n)})| \le \mathbb{P}(T > n)$.
(The coupling ineq.)

$d_{TV}(\nu, \nu') = \max_A \nu(A) - \nu'(A)$

For the other direction, it is possible to show that there exists a coupling for which equality is attained above for all n (maximal coupling).

Examples: 1) $\mathbb{Z}^d$: Tail σ-field is only parity and invariant field is trivial

Later, extensions to sub-linear and non-nbs

proof: coupling from two prev. page shows that if x and y have same parity then $h(x,m) = h(y,m)$ since there exists a successful coupling. Hence h is just parity.

Exercise: prove tail is just parity using a coupling

Similarly, $\forall x,y$ $\exists$ successful shift-coupling.

2) An irreducible recurrent chain with all states of period d. (gcd of possible return times is d)

$T = \{$cyclic class of starting point$\}$, $\mathcal{I}$ trivial (we proved this last statement last time) (can get from any state to any state) (period 1)

(Orey) see Durrett for a proof.

3) Long range 1D RW on $\mathbb{Z}$, Ornstein's coupling: If S_n is a 1D irreducible aperiodic RW on $\mathbb{Z}$ then T is trivial. Fix $x, y \in \mathbb{Z}$ and let $(X_n)_{n\ge 0}$ be a seq. of IID incr.

Take M large enough so that the walk with incr. $(X_1 | X_1 \le M)$ is still irreducible and aperiodic. Let $(X_n^M)_{n\ge 0}$ be ind. of (X_n) and IID dist. as $(X_1 | X_1 \le M)$.

Set $S_n^x = S_{n-1}^x + X_{n-1}$.

Now couple S_n^x and S_n^y by: If $S_{n-1}^x = S_{n-1}^y$ then $S_n^x = S_n^y$.
Otherwise, if $X_n > M$ then $S_n^y = S_{n-1}^y + X_{n-1}$. If $X_n \le M$ then $S_n^y = S_{n-1}^y + X_{n-1}^M$ (big jumps together and small jumps ind.).
Since $(S_n^y - S_n^x)_{n\ge 0}$ is a 1D RW on $\mathbb{Z}$, started at $x-y$, with mean 0, and irreducible (check!) it follows that it will hit 0 eventually.
Hence this is a successful coupling.

needed? (regular)
4) RW on a tree: $\mathcal{I}$ is the σ-field generated by the events of going to infinity on a given end. $\mathcal{T}$ adds to that the parity of the starting point.

Kaim. Vershik, Lawler.
Kaimanovich-Vershik thm (1983) RW $(S_n)_{n\ge 0}$ on the Cayley graph of a (discrete, countable) group with finite entropy increments has no non-constant bdd harmonic fcns. iff $h(G,\mu) := \lim_{n\to\infty} \frac{H(S_n)}{n} = 0$
where $H(\mu) := -\sum_x \mu^n(x)\log(\mu^n(x))$ (with $0\log(0) := 0$)

5) Lamplighter groups: $\mathbb{Z}_2 \wr G$: Cayley graph indexed by $(g, f: G \to \mathbb{Z}_2)$. (lamplighter jumps)
Liouville iff G is recurrent.
If G is transient, limit state of lamp at $e \in G$ is an invariant event.
If G is recurrent, $\frac{R_n}{n} \to 0$ by KSW thm. (R_n ← range of SRW on G)
Check! From this one can deduce that $H(S_n) = o(n)$ on $\mathbb{Z}_2 \wr G$.
So finish by Kaimanovich-Vershik.

Additional characterization: SRW on Cayley graph Liouville $\Leftrightarrow \mathbb{E}\, d(S_n, o) = o(n)$

Open questions: 1) Is Liouville a group prop. (preserved under change of gen. in Cayley graph)?
2) Possible rates of escape for SRW on groups? are they preserved under change of gen.?

Lecture 7 11.4.11 Scribe

Harmonic functions on sets in $\mathbb{Z}^d$. LCLT in $\mathbb{Z}^d$.
Green's fcns in $\mathbb{Z}^d$ and their asymptotics. Number of values seen by a RW before a geom. time (variant of KSW)
Martin capacity for Markov chains.
Intersection of RW's
Support of harmonic measure in $\mathbb{Z}^d$ (paper of Itai)

Introduction to Potential theory (Lawler-Limic, Lawler)

1. Harmonic fcns. on $\mathbb{Z}^d$ and the Dirichlet Problem

Accumulate notation on side-board

$\frac{1}{d}\sum_{i=1}^{d} \nabla_i^2 f(x) = \frac{1}{d}\sum_{i=1}^{d} \frac{1}{2}f(x+e_i)+\frac{1}{2}f(x-e_i)-f(x)$ SRW

For $f:\mathbb{Z}^d \to \mathbb{R}$, define $(\mathcal{L}f)(x) = \frac{1}{2d}\sum_{y\sim x}[f(y)-f(x)] = \left[\frac{1}{2d}\sum_{y\sim x} f(y)\right] - f(x) = \mathbb{E}^x f(S_1) - f(x)$

the discrete Laplacian. f is harmonic iff $(\mathcal{L}f)(x) = 0$. subharmonic iff $(\mathcal{L}f)(x) \geq 0$
(generator of the walk)

For $A \subseteq \mathbb{Z}^d$, let $\tau_A = \min(n \geq 1 / S_n \notin A)$, $\bar{\tau}_A = \min(n \geq 0 / S_n \notin A)$,

$\partial A = \{y \in \mathbb{Z}^d - A \mid y \sim x$ for some $x \in A\}$ the outer bdry. of A. $\bar{A} := A \cup \partial A$.

note if $f: \bar{A} \to \mathbb{Z}^d$ then $(\mathcal{L}f)(x)$ is well-defined on A.

Thm: suppose $\emptyset \neq A \subseteq \mathbb{Z}^d$ satisfies $P^x(\tau_A < \infty) = 1$ $\forall x \in A$ and let $F: \partial A \to \mathbb{R}$

Other RP and operators $\mathcal{L}$ are also pos.

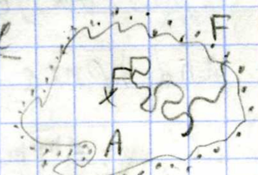
be a bdd. fcn.. Then $\exists!$ bdd. fcn. $f: \bar{A} \to \mathbb{R}$ s.t.

$$(\mathcal{L}f)(x) = 0 \ \forall x \in A \quad \text{and} \quad f(x) = F(x) \ \forall x \in \partial A.$$

It is given by $f(x) = \mathbb{E}^x F(S_{\bar{\tau}_A})$. (*).

Proof: It is simple that (*) satisfies the requirem. by the Markov prop.
Now if f is a bdd. fcn. satisfying the requirements
then $M_n := f(S_{n\wedge\bar{\tau}_A})$ is a bdd. martingale for any starting point.
By the optional sampling thm. $f(x) = \mathbb{E}^x M_{\bar{\tau}_A} = \mathbb{E}^x F(S_{\bar{\tau}_A})$.

Remarks: 1) Note the solution satisfies the maximum principle
$\sup_{x\in A} |f(x)| = \sup_{x\in\partial A} |F(x)|$



2) If A is finite, existence and uniqueness follow by noting there are $|A|$ equations in $|A|$ unknowns and the max. principle holds by def. of $\mathcal{L}$.

3) If $|A| = \infty$ there can be inf. many unbdd. sol.. E.g. $A = \{1, 2, \ldots\}$, $F(0) = 0$ and $f(x) = bx$ for some $b \in \mathbb{R}$.

4) If $d = 1, 2$, the thm. holds $\forall A$ by recurrence.
On the other hand, if $d \geq 3$ and $\exists x \in A$ s.t. $P^x(\bar{\tau}_A = \infty) > 0$ then
$f(x) = P^x(\bar{\tau}_A = \infty)$ is another bdd. sol. for the boundary data $F \equiv 0$.

5) Continuous analogue: If $D = \{x \in \mathbb{R}^d / |x| < 1\}$ (or another "nice" bdd. domain) and $f: \bar{D} \to \mathbb{R}$ is cont. on $\bar{D}$, C^2 in D and $\Delta f(x) = 0$ $\forall x \in D$ then $f(x) = \mathbb{E}^x f(B_T)$ for a standard BM $\{B_t\}_{t\geq 0}$ started at x and stopped at time $T = \min\{t \mid |B_t| = 1\}$.

For $x \in D$, the dist. of B_T (harmonic measure) has a density wrt. Leb. meas. on ∂D and this density is the Poisson kernel: $h(x, z) = c_d(1-|x|^2)/|x-z|^d$.
Thus $f(x) = c_d \int_{\partial D} f(z) \frac{1-|x|^2}{|x-z|^d} \, dS(z)$ as can be verified directly.

Two facts follow from this repr.: deriv. estimate: $\forall k$ $\exists c_k < \infty$ (ind. of f!) s.t. $\|D^k f\|_\infty \leq c_k \|f\|_\infty$ for a k'th order deriv. D^k.
Harnack ineq.: $\forall r$ $\exists c_r < \infty$ s.t. if $f \geq 0$ then $f(x) \leq c_r f(y)$ $\forall x, y$ with $|x|, |y| \leq r$.

In other words $\max_{|x|\le r} f(x) \le C_r \min_{|x|\le r} f(x)$ for a C_r ind. of f!

Discrete analogs of these Prop. hold but we will not prove them.

What if $\mathbb{P}^x(\overline{\tau}_A=\infty)$ for some $x\in A$? We will now show that the space of bdd. sol. is exactly 1-dimensional.

Thm: $\phi\neq A\subsetneq \mathbb{Z}^d$, $F:\partial A\to\mathbb{R}$ bdd. Then the only bdd. $f:\overline{A}\to\mathbb{R}$ s.t.

S sublinear F is suff. also for this repr. in $d\ge 3$.

$(\mathcal{L}f)(x)=0\ \forall x\in A$, $f(x)=F(x)\ \forall x\in\partial A$

are of the form $f(x)=\mathbb{E}^x F(S_{\overline{\tau}_A}1_{(\overline{\tau}_A<\infty)})+b\,\mathbb{P}^x(\overline{\tau}_A=\infty)$

for some $b\in\mathbb{R}$. Can also write lazy RW here

Proof: Let $\tilde{S}_n$ be a lazy SRW. That is $\mathbb{P}(\tilde{X}_1=0)=\frac{1}{2}$, $\mathbb{P}(\tilde{X}_1=\pm e_i)=\frac{1}{4d}$.

By the prev. thm, we may assume $\exists x\in A$ s.t. $\mathbb{P}^x(\overline{\tau}_A=\infty)>0$.

Let f be a bdd. sol. Then $M_n:=f(\tilde{S}_{n\wedge\overline{\tau}_A})$ is a bdd. mart. Thus

$$f(x)=\mathbb{E}^x M_n=\mathbb{E}^x f(\tilde{S}_{n\wedge\overline{\tau}_A})=\underset{\text{for any bdd. cont. of } f}{\mathbb{E}^x f(\tilde{S}_n)}-\mathbb{E}^x f(\tilde{S}_n)1_{(\overline{\tau}_A<n)}+\mathbb{E}^x F(S_{\overline{\tau}_A}1_{(\overline{\tau}_A<n)})$$

Since we have a successful coupling of $\tilde{S}_n$ under $\mathbb{P}^x$ and $\mathbb{P}^y$ $\forall x,y$ we deduce $|\mathbb{E}^x f(\tilde{S}_n)-\mathbb{E}^y f(\tilde{S}_n)|\le 2\|f\|_\infty \mathbb{P}(T>n)\xrightarrow[n\to\infty]{}0$

Thus $|f(x)-f(y)|\le 2\|f\|_\infty(\mathbb{P}^x(\overline{\tau}_A<\infty)+\mathbb{P}^y(\overline{\tau}_A<\infty))$.

Exercise: Let $U_\varepsilon=\{x\in A \mid \mathbb{P}^x(\overline{\tau}_A=\infty)\ge 1-\varepsilon\}$. Exercise: $U_\varepsilon\neq\phi$ $\forall\varepsilon\in(0,1)$. since $\exists x\in A$ $\mathbb{P}^x(\overline{\tau}_A=\infty)>0$.

Thus $\forall x,y\in U_\varepsilon$, $|f(x)-f(y)|\le 4\varepsilon\|f\|_\infty$

and hence $\exists b=\lim_{n\to\infty} f(x_n)$ for any $x_n\in U_{1/n}$ s.t. $|f(x)-b|\le 4\varepsilon\|f\|_\infty$ $\forall\varepsilon$ and $x\in U_\varepsilon$.

Exercise: Let $\rho_\varepsilon=\min(n \mid \tilde{S}_n\in U_\varepsilon)$. Exercise (related to prev. one) $\mathbb{P}^x(\overline{\tau}_A\wedge\rho_\varepsilon<\infty)=1$ $\forall\varepsilon$ and $x\in A$.

Then $\forall x\in A$, $f(x)=\mathbb{E}^x f(\tilde{S}_{\overline{\tau}_A\wedge\rho_\varepsilon})=\mathbb{E}^x f(\tilde{S}_{\rho_\varepsilon})1_{(\overline{\tau}_A>\rho_\varepsilon)}+\mathbb{E}^x F(S_{\overline{\tau}_A}1_{(\overline{\tau}_A\le\rho_\varepsilon)})$.

By bounded conv. $\lim_{\varepsilon\to 0}\mathbb{E}^x F(\tilde{S}_{\overline{\tau}_A})1_{(\overline{\tau}_A\le\rho_\varepsilon)}=\mathbb{E}^x F(S_{\overline{\tau}_A})1_{(\overline{\tau}_A<\infty)}$

(Since $U_\varepsilon\to\infty$ as $\varepsilon\to 0$ since $\mathbb{P}^x(\overline{\tau}_A=\infty)<1$ $\forall x\in A$)

Finally $f(S_{\rho_\varepsilon})1_{(\overline{\tau}_A>\rho_\varepsilon)}\to b1_{(\overline{\tau}_A=\infty)}$ by def. and prop of b.

Hence by bdd. conv. $\lim_{\varepsilon\to 0}\mathbb{E}^x f(S_{\rho_\varepsilon})1_{(\overline{\tau}_A>\rho_\varepsilon)}=b\,\mathbb{P}^x(\overline{\tau}_A=\infty)$.

Remark: b is the boundary data "at infinity". The fact we have only one such number is the same as saying the Poisson boundary is trivial. Repr. like this are the reason for the name Poisson boundary.

Def.: Poisson kernel $H_A:\overline{A}\times\partial A\to[0,1]$, $H_A(x,y)=\mathbb{P}^x(\overline{\tau}_A<\infty, S_{\overline{\tau}_A}=y)$

and also $H_A(x,\infty)=\mathbb{P}^x(\overline{\tau}_A=\infty)$ (slightly abusing notation)

Then writing $F(\infty)=b$ we have shown $f(x)=\sum_{y\in\partial A\cup\{\infty\}} H_A(x,y)F(y)$.

We turn now to the discrete Poisson equation.

Thm: $\emptyset \neq A \subsetneq \mathbb{Z}^d$. $g: A \to \mathbb{R}$ with finite support. Then the unique bdd. fcn f on $\bar{A}$ vanishing on ∂A and satisfying $(\mathcal{L}f)(x) = -g(x)$ $\forall x \in A$ is

need $\mathbb{P}_x(\bar{\tau}_A < \infty) = 1$ $\forall x \in A$.

$$f(x) = \mathbb{E}^x\left(\sum_{j=0}^{\bar{\tau}_A - 1} g(S_j)\right) = \sum_{y \in A} G_A(x,y) g(y) \text{ with } G_A(x,y) = \mathbb{E}^x\left(\sum_{j=0}^{\bar{\tau}_A - 1} 1_{(S_j = y)}\right) =$$ number of visits to y starting at x before leaving A

G_A is the Green's fcn for A.

Remarks: For finite A, G_A is a matrix and defining $\mathcal{L}_A(x,y) = \begin{cases} -1 & x = y \\ \frac{1}{2d} & x \sim y \\ 0 & x \neq y, x \not\sim y \end{cases}$

then the thm. shows $G_A = -(\mathcal{L}_A)^{-1}$.

2) $\mathbb{E}^x(\bar{\tau}_A) = \sum_{y \in A} G_A(x,y)$ by taking $g \equiv 1$. This is the unique bdd. fcn. $f: \bar{A} \to \mathbb{R}$ vanishing on ∂A and having $\mathcal{L}f \equiv -1$ on A.

$\mathcal{L}_A = \begin{pmatrix} -1 & \frac{1}{2} & \\ \frac{1}{2} & -1 & \frac{1}{2} \\ & \ddots & \end{pmatrix}$ in $d = 1$ on an interval A.

Proof: It is easy to check this f satisfies the requirements.

exercise

For other dir., exercise: $\forall f: \mathbb{Z}^d \to \mathbb{R}$, $M_n := f(S_n) - \sum_{j=0}^{n-1} (\mathcal{L}f)(S_j)$ is a martingale.

Let f be a bdd. sol. We deduce $M_n := f(S_{n \wedge \bar{\tau}_A}) + \sum_{j=0}^{n \wedge \bar{\tau}_A - 1} \underbrace{(\mathcal{L}f)(S_j)}_{\shortparallel\, g}$

is a martingale. Noting $|M_n| \le \|f\|_\infty + Y$ with $Y = \sum_{j=0}^{\bar{\tau}_A - 1} |g(S_j)|$

we have $\mathbb{E}^x Y = \sum_y G_A(x,y)|g(y)| < \infty$ and thus by the dom. conv. criterion for optional sampling: $f(x) = \mathbb{E}^x M_{\bar{\tau}_A} = \mathbb{E}^x\left(\sum_{j=0}^{\bar{\tau}_A - 1} g(S_j)\right)$ as required.

Remark: To get a sol. not vanishing on ∂A we add a sol. of the Dirichlet Problem.

2. Local central limit theorem

Recall that if $X \sim N(0,n)$ then its density is $\frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}}$.

We have shown that for 1D SRW, for even $n+k$ and $k = o(n^{3/4})$

we have $\mathbb{P}(S_n = k) \sim 2 \cdot \frac{1}{\sqrt{2\pi n}} e^{-k^2/2n}$. And we could get more explicit error estimates.

Now for large n, S_n is roughly $N(0, \frac{n}{d} I)$ (cov. matrix) (since e.g. $\mathbb{E} X_{1,1}^2 = \frac{1}{d}$)

If $X \sim N(0, \frac{n}{d} I)$ then its density is $\left(\frac{1}{\sqrt{2\pi \frac{n}{d}}}\right)^d e^{-\frac{d|x|^2}{2n}} = \left(\frac{d}{2\pi n}\right)^{d/2} e^{-\frac{d|x|^2}{2n}}$.

Thus we may expect that if $k \in \mathbb{Z}^d$ and n have the same parity and $|k|$ is small enough then $\underbrace{\mathbb{P}(S_n = k)}_{p_n(k)} \approx 2 \cdot \left(\frac{d}{2\pi n}\right)^{d/2} e^{-\frac{d|k|^2}{2n}} =: \bar{p}_n(k)$

Let $E(n,k) = \begin{cases} |p_n(k) - \bar{p}_n(k)| & k \text{ and } n \text{ have same parity} \\ 0 & \text{otherwise} \end{cases}$

Thm. (LCLT): $\sup_k |E(n,k)| \le O(n^{-(d+2)/2})$ as $n \to \infty$.

We will not prove this now (possibly later?). There are two main approaches to the proof:

1) Note the vector $(n_1, \dots, n_d)$ of the number of moves in each coord. has a multinomial dist., show a local limit thm for it and rely on the 1D results in each coord.
This is easier in cont. time, when we make a Poisson number of moves since then coord. are ind.

2) Write the Fourier transforms of $P_n(k)$ and $\bar{P}_n(k)$ and note that $\|\underbrace{P_n(k)-\bar{P}_n(k)}_{\substack{0 \text{ when } n \text{ and} \\ k \text{ have diff. parity}}}\|_\infty \le \|\hat{P}_n(k)-\hat{\bar{P}}_n(k)\|_1$. Evaluate by Taylor expansion near origin and by triangle ineq. elsewhere.

Prop. (large dev.): $\exists c_1, c_2 > 0$ s.t. $\mathbb{P}(|S_n| \ge r\sqrt{n}) \le c_2 e^{-c_1 r^2}$ $\forall r, n$

Proof: Follows from 1D estimate we proved: $\mathbb{P}(S_n \ge k) \le e^{-k^2/2n}$ in 1D $\forall k, n$.

3. Green's fcns.

There are 3 types of Green's fcns. commonly discussed:

1) $G(x,y) = \mathbb{E}^x(\#\text{visits to } y) = \sum_{n=0}^{\infty} \underbrace{\mathbb{P}^x(S_n=y)}_{P_n(y-x)}$ defined only for $d \ge 3$
$G(x,y) =: G(y-x)$ since we need transience.
For $d=1,2$, the potential kernel $a(x) = \sum_{n=0}^{\infty}(P_n(0)-P_n(x)) = "G(0)-G(x)"$ plays a similar role as $-G(x)$.

2) $\underbrace{G(x,y;\lambda)}_{G(y-x;\lambda)}$: Let T be a $\overset{\text{ind.}}{}$ geometric "killing time" with parameter $1-\lambda$ (chance to be killed at every step). $\mathbb{P}(T>n) = \lambda^n$.
Then $G(x,y,\lambda) = \mathbb{E}^x \#$ of visits to y before time $T = \sum_{n=0}^{\infty} \mathbb{P}^x(S_n=y)\lambda^n$.

3) $G_A(x,y) = \mathbb{E}^x(\#$ of visits to y before leaving $A)$.

These are more useful than, say, number of visits to y by some fixed time N since they are "Markovian". After n steps, we are in same situation for each of them.
Their asymptotics can, in case 1, 2, be deduced from the LCLT and large dev. results.

Application: range of 1D and 2D RW: Consider a geometrically killed SRW as above. Let $R = R^\lambda$ be the number of distinct positions it visited. Then, by a variant of the KSW argument, we have $\mathbb{E}R \overset{\text{Exercise}}{=} \mathbb{E}T \cdot \mathbb{P}(\forall n \ge 1, S_n \ne 0) = \frac{1}{1-\lambda}\mathbb{P}(\text{no return to } 0)$.
However, since the number of returns to 0 is geometric, we have $\mathbb{P}(\text{no return to } 0) = G(0;\lambda)^{-1}$

Finally, $G(0;\lambda)=\sum_{n=0}^{\infty}\lambda^n p_n(0)=\sum_{n=0}^{\infty}\lambda^n\left(\bar{p}_n(0)+O\left(\frac{1}{n^{(d+2)/2}}\right)\right)=$

$=\sum_{n=0}^{\infty}\lambda^n\left(\frac{c_d}{n^{d/2}}+O\left(\frac{1}{n^{(d+2)/2}}\right)\right)\underset{\uparrow}{\approx} c_d F\left(\frac{1}{1-\lambda}\right)$ where $F(s)=\begin{cases}\sqrt{s} & d=1\\ \log s & d=2\end{cases}$

as $\lambda\to 1$, exercise

(we have $c_d=\left(\frac{d}{2\pi}\right)^{d/2}\cdot\begin{cases}\Gamma(1/2) & d=1\\ 1 & d=2\end{cases}$)

Thus if $\lambda=1-\frac{1}{n}$ (about n steps) then $\mathbb{E}R\sim\begin{cases}\frac{n}{c_1\sqrt{n}}=\frac{\sqrt{n}}{c_1} & d=1\\ \frac{n}{c_2\log n} & d=2\end{cases}$

In particular we may deduce the non-trivial result that an n-step 2D SRW visits $\frac{n}{c_2\log n}$ vertices on average (quantitative recurrence) or BSW

Lecture 8 29.4.11 (makeup class) Scribe

Reminder of LCLT: $\sup_{\substack{x\in\mathbb{Z}^d\\ n,x \text{ of same parity}}}\Big|\overbrace{\mathbb{P}(S_n=x)-2\left(\frac{d}{2\pi n}\right)^{d/2}e^{-d|x|^2/2n}}^{E(n,x)}\Big|\le O\left(n^{-(d+2)/2}\right)$ $(n\to\infty)$

Large dev. prop.: $\exists C_d, c_d>0,\ \forall r,n$ $\mathbb{P}(|S_n|\ge r\sqrt{n})\le C_d e^{-c_d r^2}$

Reminder of types of Green's fcn's, correspond to types of transient walks:

1) Whole space: $G(x,y)=\mathbb{E}^x(\text{visits to } y)=\sum_{n=0}^{\infty}\mathbb{P}^x(S_n=y)=G(x-y)$

In $d=1,2$, potential kernel $a(x)=\sum_{n=0}^{\infty}\mathbb{P}(S_n=0)-\mathbb{P}(S_n=x)=\text{"}G(0)-G(x)\text{"}$.

2) Killed walk: $T\sim\exp(1-\lambda)$ ind. of walk. $G(x)=\sum_{n=0}^{\infty}\mathbb{P}(S_n=x,\ T>n)$

3) Domain Green's fcn: $A\subseteq\mathbb{Z}^d$, $G_A(x,y)=\mathbb{E}^x(\#\text{visits to } y \text{ before leaving } A)$.
$=\sum_{n=0}^{\infty}\mathbb{P}^x(S_n=y,\ \underbrace{\tau_A}_{\min(n\ge 0\,|\,S_n\notin A)}>n)$

Lemma: $G_A(x,y)=G_A(y,x)\ \forall x,y\in A$.

Proof: By traversing paths backward
$\mathbb{P}^x(S_n=y,\tau_A>n)=\mathbb{P}^y(S_n=x,\tau_A>n)$.

Asymptotics for Green's fcn's

Thm.: For $d\ge 3$, $G(x)\sim a_d|x|^{2-d}$ as $x\to\infty$, with $a_d=\frac{d}{2}\Gamma\left(\frac{d}{2}-1\right)\pi^{-d/2}=\frac{2}{(d-2)\omega_d}$ — $|x|$ is Euclidean norm

More precisely: $\forall\alpha<d$, $G(x)=a_d|x|^{2-d}+o(|x|^{-\alpha})$ $x\to\infty$. ($\omega_d=\mathrm{Vol}(B(0,1))$ in $\mathbb{R}^d$.

Proof: Fix $x\neq 0$ of even parity. Let $\bar{p}(2n,x)=2\left(\frac{d}{4\pi n}\right)^{d/2}e^{-d|x|^2/4n}$.

$\sum_{n=1}^{\infty}\bar{p}(2n,x)=\sum_{n=1}^{\infty}2\left(\frac{d}{4\pi n}\right)^{d/2}e^{-d|x|^2/4n}\underset{\text{exercise}}{=}\int_0^{\infty}2\left(\frac{d}{4\pi t}\right)^{d/2}e^{-d|x|^2/4t}dt+O(|x|^{-d})=$

$=\frac{d}{2}\Gamma\left(\frac{d}{2}-1\right)\pi^{-d/2}|x|^{2-d}+O(|x|^{-d})$, as $x\to\infty$.

Now $G(x)=\sum_{n=0}^{\infty}\mathbb{P}(S_{2n}=x)=\sum_{n=0}^{\infty}\bar{p}(2n,x)+\sum_{n=0}^{\infty}E(2n,x)$

It remains to show that $\forall\alpha<d$, $\sum_{n=0}^{\infty}|E(2n,x)|=o(|x|^{-\alpha})$ as $x\to\infty$

This follows by considering separately $n\ll x^2$ and $n\gg x^2$ and using the large dev. estimate in the first regime and the LCLT in the second.

Remark: In the continuum, for BM, $G(x)=a_d|x|^{2-d}$.
It is a harmonic fcn. $\forall x\neq 0$ and has $-\delta_0$ as laplacian at 0.
For x of odd parity we get the result since G is harmonic off 0, by averaging over the neighbors of x.

Remark: It is actually true that the error term is $O(|x|^{-d})$.

$A_{n,m}$ is better

Application (exiting annuli and transience): $(d\geq 3)$ For $n<m$, let $A=\{x \mid n<|x|<m\}$ and $\tau=\min(j \mid S_j\notin A\}$, then for $x\in A$, $\mathbb{P}^x(|S_\tau|\leq n)=\frac{|x|^{2-d}-m^{2-d}+O(n^{1-d})}{n^{2-d}-m^{2-d}}$ as $n\to\infty$

Note $\mathbb{P}^x(|S_\tau|\leq n)=\left(\frac{|x|}{n}\right)^{2-d}+O(n^{-1})$ as $n\to\infty$. Quantitative transience.

Proof: $G(x)$ is harmonic off 0, hence $M_j=G(S_{j\wedge\tau})$ is a bdd. mart.
By optional sampling $G(x)=\mathbb{E}^x(M_\tau)=\mathbb{P}^x(|S_\tau|\leq n)\mathbb{E}^x(G(S_\tau)\mid |S_\tau|\leq n)+$
$+(1-\mathbb{P}^x(|S_\tau|\leq n))\mathbb{E}^x(G(S_\tau)\mid |S_\tau|\geq m)$

and the result follows from the prev. thm. using the weaker estimate $G(x)=a_d|x|^{2-d}+O(|x|^{1-d})$.

Similarly, letting $C_n=\{x \mid |x|<n\}$ and $\eta=\min(j \mid S_j\in\{0\}\cup C_n^c)$
then for $x\in C_n$, $\mathbb{P}^x(S_\eta=0)=\frac{a_d}{G(0)}(|x|^{2-d}-n^{2-d})+O(|x|^{1-d})$ (*)

To calculate Green's fcn. in a domain, we have the following

Prop.: For $d\geq 3$, and finite $A\subseteq\mathbb{Z}^d$, $\forall x,z\in A$,
$$G_A(x,z)=G(z-x)-\sum_{y\in\partial A}H_{\partial A}(x,y)G(z-y)$$
$\partial A=\{y\notin A \mid y\sim w\in A\}$

If $\tau=\min(n \mid S_n\notin A)$, $H_{\partial A}(x,y)=\mathbb{P}(S_\tau=y)$

Proof: $G_A(x,z)=\mathbb{E}^x\left(\sum_{j=0}^{\tau-1}1(S_j=z)\right)=\mathbb{E}^x\left(\sum_{j=0}^{\infty}1(S_j=z)-\sum_{j=\tau}^{\infty}1(S_j=z)\right)$.

Cor.: $G_{C_n}(x,0)=a_d(|x|^{2-d}-n^{2-d})+O(|x|^{1-d})$

Proof: $G_{C_n}(x,0)=\mathbb{P}^x(S_\eta=0)G_{C_n}(0,0)$, $G_{C_n}(0,0)=G(0)+O(n^{2-d})$ and (*)

Dimensions 1,2: The analogous results for $d=1,2$ are:

For $d=1$, $a(x)=|x|$

For $d=2$, $\exists k$ s.t. $\forall\alpha<2$ $a(x)=\frac{2}{\pi}\ln|x|+k+O(|x|^{-\alpha})$

(and $k=\frac{2\gamma}{\pi}+\frac{3}{\pi}\ln 2$, $\gamma=$ Euler's const. $=\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j}-\ln(n)\right)$)

$G_A(x,z)\underset{d=2}{=}\sum_{y\in\partial A}H_{\partial A}(x,y)a(y-x)-a(z-x)$ for finite A, $x,z\in A$.

For $A=\{x \mid n<|x|<m\}$, $\tau=\min(j \mid S_j\notin A)$, $x\in A$,
$$\mathbb{P}^x(|S_\tau|\leq n)=\frac{\ln m-\ln|x|+O(n^{-1})}{\ln m-\ln n}.$$
Quantitative recurrence

and for $d=2$ $\forall \alpha<2$ $G_{C_n}(x,0)=\frac{2}{\pi}(\ln n-\ln|x|)+o(|x|^{-\alpha})+O(n^{-1})$.

(important for discrete Gaussian free field, say)

<u>Martin capacity for Markov chains</u> (following Benjamini-Pemantle-Peres 1995)

<u>Def.:</u> Given a measure space $(\Lambda,\mathcal{F})$, a measurable function (called kernel) $F:\Lambda\times\Lambda\to[0,\infty]$ and a finite measure μ on Λ, the <u>F-energy</u> of μ is $I_F(\mu)=\int_\Lambda\int_\Lambda F(x,y)\,d\mu(x)\,d\mu(y)$

The <u>capacity of Λ wrt. F</u> is $\mathrm{Cap}_F(\Lambda)=(\inf_\mu I_F(\mu))^{-1}$ with $\infty^{-1}:=0$, where the infimum is over all prob. meas. μ on F.

For us, Λ will be countable and $\mathcal{F}$ will be all subsets, but these ideas work equally well in $\mathbb{R}^d$.

Furthermore define the <u>asymptotic capacity</u> of Λ wrt. F by $\mathrm{Cap}_F^{(\infty)}(\Lambda):=\inf_{\Lambda_0\text{ finite}}\mathrm{Cap}_F(\Lambda\setminus\Lambda_0)$. ← note capacity is monotone in the set.

Now let $\{X_n\}_{n=0}^\infty$ be a Markov chain on a set Y with transition prob. $p(x,y)$ and define its Green's fcn.:

$G(x,y):=\sum_{n=0}^\infty \mathbb{P}^x(X_n=y)=\sum_{n=0}^\infty p^{(n)}(x,y)$.

We want to estimate the prob. that X_n hits a set $\Lambda\subseteq Y$.

<u>Thm.:</u> Let $\{X_n\}$ be a transient Markov chain (no state is visited i.o.) on a countable state space Y and $\Lambda\subseteq Y$. then $\forall \rho\in Y$:

$$\tfrac{1}{2}\mathrm{Cap}_K(\Lambda)\le \mathbb{P}^\rho(n\ge 0,\ X_n\in\Lambda)\le \mathrm{Cap}_K(\Lambda) \quad (1)$$

$$\tfrac{1}{2}\mathrm{Cap}_K^{(\infty)}(\Lambda)\le \mathbb{P}^\rho(X_n\in\Lambda\text{ i.o.})\le \mathrm{Cap}_K^{(\infty)}(\Lambda) \quad (2)$$

where K is the Martin kernel: $K(x,y):=\frac{G(x,y)}{G(\rho,y)}$ defined using ρ.

<u>Remarks:</u> 1) Historically, the Green's fcn. itself is used as the kernel for SRW or BM. However, note that the Green's fcn. is translation-inv. whereas the hitting prob. are not. Furthermore, for BM, hitting prob. are scale-inv. whereas the Green's fcn. is not. The Martin kernel remedies both issues.

2) It suffices instead of transience to assume $G(x,y)<\infty$ $\forall x,y\in\Lambda$.

3) The Martin kernel can be replaced by its sym. version $\frac{1}{2}(K(x,y)+K(y,x))$ without affecting the capacities.

4) The thm. extends to having init. dist. π by adding an abstract state ρ which moves according to π on the first step.

Proof: The right hand ineq. follows by taking μ to be the harmonic measure. That is, let $\tau = \min(n \geq 0 / X_n \in \Lambda)$ and $\nu(x) = \mathbb{P}^\rho(X_\tau = x, \tau < \infty)$

$\forall y \in \Lambda \quad \int G(x,y)\, d\nu(x) = \sum_{x \in \Lambda} \mathbb{P}^\rho(X_\tau = x) G(x,y) = G(\rho, y)$. if $\nu(\Lambda) = 0$ the right hand ineq. is trivial.

Thus $\int K(x,y)\, d\nu(x) = 1 \quad \forall y \in \Lambda$. Consequently:

$$I_K\left(\frac{\nu}{\nu(\Lambda)}\right) = \nu(\Lambda)^{-2} I_K(\nu) = \nu(\Lambda)^{-1} \Rightarrow \mathrm{Cap}_K(\Lambda) \geq \nu(\Lambda).$$

For the left hand ineq., we use the second moment method. Given a prob. measure μ on Λ, let

$$Z = \int_\Lambda G(\rho, y)^{-1} \sum_{n=0}^{\infty} 1_{(X_n = y)}\, d\mu(y).$$

Using Fubini, $\mathbb{E}_\rho Z = 1$. We bound the second moment

$$\mathbb{E}_\rho Z^2 = \mathbb{E}_\rho \int_\Lambda \int_\Lambda G(\rho, y)^{-1} G(\rho, x)^{-1} \sum_{m,n=0}^{\infty} 1_{(X_m = x, X_n = y)}\, d\mu(x)\, d\mu(y) \leq$$

$$\leq 2\mathbb{E}_\rho \int_\Lambda \int_\Lambda G(\rho, y)^{-1} G(\rho, x)^{-1} \sum_{0 \leq m \leq n < \infty} 1_{(X_m = x, X_n = y)}\, d\mu(x)\, d\mu(y)$$

For each m, we have $\mathbb{E}_\rho \sum_{n=m}^{\infty} 1_{(X_m = x, X_n = y)} = \mathbb{P}_\rho(X_m = x) G(x,y)$

Therefore, $\mathbb{E}_\rho Z^2 \leq 2 \int_\Lambda \int_\Lambda G(\rho, y)^{-1} G(x,y)\, d\mu(x)\, d\mu(y) = 2 I_K(\mu)$

By the second moment ineq. (Cauchy-Schwartz),

$$\mathbb{P}_\rho(\exists n, X_n \in \Lambda) \geq \mathbb{P}_\rho(Z > 0) \geq \frac{(\mathbb{E}_\rho Z)^2}{\mathbb{E}_\rho(Z^2)} \geq \frac{1}{2 I_K(\mu)}$$

Maximizing over μ we have $\mathbb{P}_\rho(\exists n, X_n \in \Lambda) \geq \frac{1}{2} \mathrm{Cap}_K(\Lambda)$

To infer (2) from (1), note that since $\{X_n\}$ is transient,

$$\{X_n \in \Lambda \text{ i.o.}\} \overset{(\mathrm{mod}\ 0)}{=} \bigcap_{\Lambda_0 \text{ finite}} \{\exists n, X_n \in \Lambda \setminus \Lambda_0\}$$

then by monotonicity, if $\{\Lambda_n\}_{n \geq 0}$ is a seq. of finite sets increasing to Λ, then $\mathbb{P}(X_n \in \Lambda \text{ i.o.}) = \lim_{n \to \infty} \mathbb{P}(\exists n\ X_n \in \Lambda \setminus \Lambda_n)$

and $\mathrm{cap}_K^{\infty}(\Lambda) = \lim_{n \to \infty} \mathrm{cap}_K(\Lambda \setminus \Lambda_n)$

There are many nice corollaries in the paper of BPP. Here is one:

Cor.: Define the Riesz-type kernel $F_\alpha(x,y)=\frac{\|y\|^\alpha}{1+\|x-y\|^\alpha}$ on $\mathbb{Z}^d$ where $\|\cdot\|$ is any norm. By the prev. thm. and the Hewitt-Savage 0-1 law, SRW on $\mathbb{Z}^d$ for $d\geq 3$ visits a set Λ i.o. a.s. iff $\mathrm{Cap}^{(\infty)}_{d-2}(\Lambda)>0$.

Remark: For BM, a Borel set is called polar if $\forall x$,

Put after def. of capacity as motivation

$\mathbb{P}^x(B_t\in A \text{ for some } t>0)=0$. Kakutani (44) showed that A is polar iff $\mathrm{Cap}_f(A)=0$ for $f(x)=\begin{cases}|\log(1/|x|)| & d=2\\ |x|^{2-d} & d\geq 3\end{cases}$

The prev. thm. (with $K(x,y)=\frac{|y|^{d-2}}{|x-y|^{d-2}}$ for $d\geq 3$) can be shown to apply also to BM and to give a quantitative extension to Kakutani's thm.

Intersection of random walks (following Lawler-Limic)

Let $S[n_1,n_2]=\{S_n \mid n_1\leq n\leq n_2\}$. We have

Thm.: $\exists C_1,C_2>0, \forall n\geq 2$ $C_1\phi(n)\leq\mathbb{P}(S[0,n]\cap S[2n,3n]\neq\emptyset)\leq\mathbb{P}(S[0,n]\cap S[2n,\infty)\neq\emptyset)\leq C_2\phi(n)$

With $\phi(n)=\begin{cases}1 & d<4\\ 1/\log n & d=4\\ 1/n^{(d-4)/2} & d>4\end{cases}$.

Remarks: 1) Same holds for any sym., fin. supp. irred. walk.

2) Taking $n\to\infty$ we get $\mathbb{P}(B[0,1]\cap B[2,3]\neq\emptyset)\begin{cases}>0 & d\leq 3\\ =0 & d\geq 4\end{cases}$ for a BM B.

Proof (following Lawler-Limic) Trivial UB for $d\leq 3$. LB for $d\leq 2$ follows from $d=3$. Hence assume $d\geq 3$. We use the second moment method. Let $J_n=\sum_{j=0}^{n}\sum_{k=2n}^{3n}1(S_j=S_k)$ $K_n=\sum_{j=0}^{n}\sum_{k=2n}^{\infty}1(S_j=S_k)$.

Using the LCLT and large dev. $C_1 n^{\frac{4-d}{2}}\leq\mathbb{E}J_n\leq\mathbb{E}K_n\leq C_2 n^{\frac{4-d}{2}}$.

Similarly, the LCLT and large dev. give $\mathbb{E}J_n^2\leq\begin{cases}cn & d=3\\ c\log n & d=4\\ cn^{(4-d)/2} & d\geq 5\end{cases}$

Since $\mathbb{P}(K_n\geq 1)\leq\mathbb{E}K_n$ and $\mathbb{P}(J_n>0)\geq\frac{\mathbb{E}(J_n)^2}{\mathbb{E}(J_n^2)}$ the lower bounds for all d and the upper bounds for all $d\neq 4$ follow.

The upper bound for $d=4$ needs more work (estimating $\mathbb{E}(K_n \mid K_n\geq 1)\geq c\log n$) and we will not do it here (see Lawler-Limic).

like expected number of intersections for two walks starting at origin

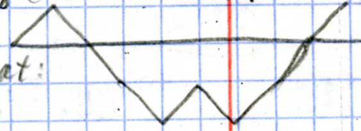
Consider explaining easy example in Lawler-Limic after proof.

Briefly overview one-sided exponent section from Lawler-Limic.

Lecture 9 2.5.11 (Scribe)

Brownian motion

Motivation: Scaling limit of RW $S_n^*(t) := \frac{S(nt)}{\sqrt{n}}$ interpolated linearly will converge weakly to BM in $C[0,1]$. (Donsker inv. Princip.)



We aim to construct $\{B(t)\}_{t\geq 0}$. Expect intuitively that:

1) Ind. increments: $\forall\, 0 \leq t_1 \leq t_2 \leq \_\_ \leq t_n$:

$B(t_n) - B(t_{n-1}), \_\_\_, B(t_2) - B(t_1)$ are ind.

2) The dist. of the increment $B(t+h) - B(t)$ does not depend on t: stationary increments.

3) The process $\{B(t)\}_{t>0}$ is a.s. continuous.

By the CLT, we can also expect that the increments are Gaussian. Our first task is to show such a process exists

Construction of BM: Def.: A real-valued stochastic process $\{B(t)\}_{t>0}$ is called a (linear) BM with start in $x \in \mathbb{R}$ if

1) $B(0) = x$ a.s.

2) The process has ind. increments.

3) $\forall t \geq 0, h > 0$, $B(t+h) - B(t) \sim N(0,h)$.

4) a.s., the function $t \mapsto B(t)$ is cont..

If $x = 0$, we call B a standard BM.

same as Borel σ-algebra of $(C[0,1])^\infty$ with product topology

More precisely, $t \mapsto B(t)$ is a random element of $C[0,\infty)$ with its Borel σ-algebra. That is, we have a common Prob. space $(\Omega, \mathcal{F}, P)$ s.t. $\omega \mapsto (t \mapsto B(t,\omega))$ is a RV in $C[0,\infty)$. Explain finite-dim. dist. and that being cont. is not in their σ-algebra. Explain that 1,2,3,4 determine the dist. of B uniquely as a random element of $C[0,\infty)$.

Construction of BM: Give some prop. of Gaussian RVs.

Def.: We say that $X=(X_1,\dots,X_n)$ is dist. as a Gaussian vector if $X \overset{d}{=} AY+b$ for $Y=(Y_1,\dots,Y_k)$ IID $N(0,1)$ and some $k\geq 1$, $A\in M_{n\times k}$ and $b\in\mathbb{R}^n$.

Thm.: The dist. of a Gaussian vector is uniq. determined by its mean vector and cov. matrix. (proof, e.g., by Fourier transform)

Cor.: If X_1, X_2 are ind. $N(0,1)$ then $\frac{X_1+X_2}{\sqrt{2}}$, $\frac{X_1-X_2}{\sqrt{2}}$ are also ind. $N(0,1)$.

Prop.: If $\{X_n\}_{n\geq 1}$ is a seq. of Gaussian vectors in $\mathbb{R}^n$ conv. in dist. to X then X is $N(\mu,\Sigma)$ with μ and Σ the limits (which exist) of the mean and cov. of X_n

Prop. (Tail estimate): If $X\sim N(0,1)$ then

$$\frac{x}{x^2+1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \leq P(X>x) \leq \frac{1}{x}\cdot\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Did existence thm. and showed upper and lower bounds for modulus of continuity.

Lecture 10 13.5.11 (make-up class)

Reminders: BM starting at x is a random cont. fcn. on $[0,\infty)$ s.t. $B(0)=x$ a.s., independent incr. and $B(t+h)-B(h)\sim N(0,h)$.

Discussed modulus of cont. of $B[0,1]$:

1) $\exists C>0$ s.t. a.s., $\forall$ suff. small h and all $0\leq t\leq 1-h$ we have $|B(t+h)-B(t)|\leq C\sqrt{h\log(1/h)}$

2) $\forall c<\sqrt{2}$, a.s. $\forall\varepsilon>0$ $\exists 0<h<\varepsilon$ and $t\in[0,1-h]$ s.t. $|B(t+h)-B(t)|\geq c\sqrt{h\log(1/h)}$.

Hölder cont.: Def.: $f:[0,\infty)\to\mathbb{R}$ is said to be locally α-Hölder cont. at $x\geq 0$ if $\exists\varepsilon>0, c>0$ s.t. $|f(x)-f(y)|\leq c|x-y|^\alpha$ $\forall y\geq 0$ with $|x-y|<\varepsilon$.

$\alpha>0$ is called the Hölder exp. and $c>0$ the Hölder const.

Cor.: $\forall\alpha<\frac{1}{2}$, a.s. BM is locally α-Hölder cont. at every point (simultaneously).

Proof: It follows immed. from the mod. of cont. results for $B[0,1]$. Hence it is also true $\forall k$ for $B_k(t):=B(t+k)-B(k)$ for $t\in[0,1]$.

- need comp. of $\mathcal{F}$ for first ex. on p. 112
- p. 24, seems like 3 on display 5 should be 2.
- p. 21 = instead of $\leq$ in l. (-7).
- p. 22 BV implies diff. a.e., no?
- p. 12, Cannot write $X_t = \frac{1}{a}B_{a^2t}$ on bottom since $a<0$.
- Missing the. p. 39, l. (-7).
- If $T=\infty$ in proof of Thm. 2.14 p. 44.

- p. 15, why 2 in bound on $\|F_n\|_\infty$?
- p. 21 $k=2^n-12$ why restrict $j\leq 2^n-k$?
- Proof of thm. 1.35?
- in proof of thm. 1.30, no need for 2^{-n} interv. and BC.
- p. 11, since def. of F_n has $2^{-(n+1)/2}$, no need for $\sqrt{2\log 2}$, can take any $c>\sqrt{2\log 2}$.
- p. 15, can also bound $|F_n(t+h)-F_n(t)|$ by $\|F_n\|_\infty$ without 2 by special def. of F_n.
- Proof of Wiener thm. 1.3. Would help to (or explain better) elaborate more on why B has ind. incr. on D_n. Maybe better to start with the F_n.
- In def. of BM, what is the σ-alg. for its space? Would help to understand whether its dist. is uniq. specified.

Remark: It is also true that for any $\alpha > \frac{1}{2}$, a.s. at every point BM is not loc. α-Hölder cont. This may be an exercise.

Exercise

Points where BM is loc. $\frac{1}{2}$-Hölder cont. exist, but are very rare (See Mörters-Peres under "slow points", we will not prove this).

Non-differentiability: Thm. (Paley, Wiener, Zygmund 1933): A.s. BM is not diff. anywhere. Moreover, $\forall t$, either $D^*B(t) = \infty$ or $D_*B(t) = -\infty$ or both, where $D^*f(t) = \limsup_{h \downarrow 0} \frac{f(t+h)-f(t)}{h}$

$$D_*f(t) = \liminf_{h \downarrow 0} \frac{f(t+h)-f(t)}{h}.$$

Proof: It suffices to prove the thm. for all $t \in [0,1]$.

If there exists $t_0 \in [0,1]$ with $-\infty < D_*B(t) \le D^*B(t) < \infty$ then $\limsup_{h \downarrow 0} \frac{|B(t+h)-B(t)|}{h} < \infty$ and hence $\exists M, t_0 \in [0,1]$ s.t. $\sup_{h \in [0,1]} \frac{|B(t_0+h)-B(t_0)|}{h} \le M$. It suffices to show this has zero prob. for every M.

Fix M and $n \in \mathbb{N}$, $n \ge 4$, and suppose $t_0 \in [\frac{k-1}{n}, \frac{k}{n}]$ for some $1 \le k \le n$. It follows that for all $1 \le j \le n-1$ we have

$$|B(\tfrac{k+j}{n}) - B(\tfrac{k-1+j}{n})| \le |B(\tfrac{k+j}{n}) - B(t_0)| + |B(\tfrac{k-1+j}{n}) - B(t_0)| \le M \cdot \frac{2j+1}{n}.$$

Let $\Omega_{n,k} := \{|B(\frac{k+j}{n}) - B(\frac{k-1+j}{n})| \le M\frac{2j+1}{n}$ for $j = 1,2,3\}$.

Then by ind. of the increments: $P(\Omega_{n,k}) = \prod_{j=1}^{3} P(|\underbrace{B(\tfrac{k+j}{n}) - B(\tfrac{k-1+j}{n})}_{N(0,\frac{1}{n})}| \le M\frac{2j+1}{n}) =$

$Z \sim N(0,1)$ $\;= \prod_{j=1}^{3} P(|Z| \le M \cdot \frac{2j+1}{\sqrt{n}}) \le P(|Z| \le \frac{7M}{\sqrt{n}})^3 \le \left(\frac{7M}{\sqrt{n}}\right)^3$

since the normal density is bdd. by $1/2$. Hence

$P(\bigcup_{k=1}^{n} \Omega_{n,k}) \le n\left(\frac{7M}{\sqrt{n}}\right)^3 \xrightarrow[n \to \infty]{} 0$. Since existence of M and t_0 as above implies that $\bigcup_{k=1}^{n} \Omega_{n,k}$ holds for all $n \ge 4$, the thm. follows.

Dist. Prop. of the BM process

Scaling inv.: If B is a standard BM and $a > 0$ the process $\{X(t) \mid t \ge 0\}$, defined by $X(t) := \frac{1}{a} B(a^2 t)$ is also a standard BM.

Proof: Need only to check that increments have correct dist.

Remark: Individual paths change, but the overall dist. does not.

Application: For $a<0<b$, let $T(a,b) := \min\{t\geq 0 \mid B(t)\in\{a,b\}\}$. (better $a,b>0$ and $\in\{-a,b\}$?)

Then letting $X(t) := \frac{1}{|a|}B(a^2t)$ we have $\mathbb{E}\,T(a,b) = a^2 \min\{t\geq 0 \mid X(t)\in\{-1,\frac{b}{|a|}\}\} = a^2\,\mathbb{E}\,T(-1,\frac{b}{|a|})$.

In particular $\mathbb{E}\,T(-b,b) = cb^2$ for some c.

Similarly $\mathbb{P}(B(T(a,b))=a) = \mathbb{P}(X(T(-1,\frac{b}{|a|}))=-1) = f(\frac{b}{|a|})$ only.

Time inversion: If B is a standard BM then $\{X(t) \mid t\geq 0\}$ defined by $X(t) = \begin{cases} tB(1/t) & t>0 \\ 0 & t=0 \end{cases}$ is also a standard BM.

Proof: Easy to check finite dim. dist. remain the same by checking cov structure. It remains only to show cont. at 0.

Since dist. on rationals is same as for BM, a.s. $\lim_{\substack{t\downarrow 0\\ t\in\mathbb{Q}}} X(t)=0$.

By cont. of X on $(0,\infty)$ we get cont. at 0.

Remark: The Ornstein-Uhlenbeck process is defined by $Y(t) := e^{-t}B(e^{2t})$ for $t\in\mathbb{R}$.

It is a stat. Markov process which is a diffusion with drift towards origin prop. to dist. from it.

One may view time inversion as the statement that $Y(t)$ is time-reversible.

Relating prop. at ∞ to prop. at 0.

Applications and further path prop.: 1) LLN: a.s., $\lim_{t\to\infty} \frac{B(t)}{t} = 0$.

Proof: For $X(t)$ the time-inv. of B (standardized), we have $\lim_{t\to\infty} \frac{B(t)}{t} = \lim_{t\to\infty} X(\frac{1}{t}) = 0$ a.s.

2) a.s., $\limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}} = \infty$ and $\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} = -\infty$. (Remark: correct gauge fcn is the law of the iter. log.)

Proof: $\forall M$, $\mathbb{P}(B(n)>M\sqrt{n}$ i.o.$)\in\{0,1\}$ since $B(n) = \sum_{i=1}^{n}(B(i)-B(i-1))$ and the Hewitt-Savage 0-1 law. Fix M. Since for any seq. of events we have $\mathbb{P}(A_n \text{ i.o.}) \geq \limsup_{n\to\infty} \mathbb{P}(A_n)$ (Fatou, since $\{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$ and the mon. conv. thm.)

then $\mathbb{P}(B(n)>M\sqrt{n} \text{ i.o.}) \geq \limsup_{n\to\infty} \mathbb{P}(B(n)>M\sqrt{n}) = \mathbb{P}(B(1)>M)>0$.

Hence $\forall M$, $\mathbb{P}(B(n)>M\sqrt{n} \text{ i.o.})=1$, proving the prop..

3) a.s. $\limsup_{h\downarrow 0} \frac{B(h)}{\sqrt{h}} = \infty$ and $\liminf_{h\downarrow 0} \frac{B(h)}{\sqrt{h}} = -\infty$.

Proof: Let $X(t)$ be the time inv. process, then $\limsup_{h\downarrow 0} \frac{B(h)}{\sqrt{h}} = \limsup_{h\downarrow 0} \sqrt{h}\,X(\frac{1}{h}) = \limsup_{t\to\infty} \frac{X(t)}{\sqrt{t}} = \infty$.

4) Letting $\tau = \inf(t>0 \mid B(t)>0)$, $\sigma = \inf(t>0 \mid B(t)<0)$ we have $\mathbb{P}(\tau=0)=\mathbb{P}(\sigma=0)=1$.

Proof: Follows from 3.

5) BM has no interval of monot.

Proof: Let $0\leq a<b$ be rational. If $[a,b]$ is an int. of mono. then the incr. of any subinterval has the same sign.

Markov prop. and Blumenthal's 0-1 law

Def: d-dim. BM, $\mathbb{P}^x$ and $\mathbb{E}^x$.

Def. Ind. of proc. (for us) is ind. of fin. dim. dist..

Thm: (Markov prop.)

Def.: Filtration, filtered prob. space and adapted.

Relevant filtrations: $\mathcal{F}^0(t) := \sigma(B(s) \mid 0 \le s \le t)$. $\mathcal{F}^+(t) := \bigcap_{s>t} \mathcal{F}^0(s)$

Thm.: $\{B(t+s) - B(s) \mid t \ge 0\}$ ind. of $\mathcal{F}^+(t)$.

Proof: Approx. points from right. Ind. passes to limit.

Thm. (Blumenthal's 0-1 law): Germ σ-alg. $\mathcal{F}^+(0)$ is trivial.

Application: Tail σ-alg.: $\mathcal{T} = \bigcap_{t\ge 0} \sigma(B(s) \mid s \ge t)$ is trivial

Proof: Time inv. and Blum. 0-1 law.

Exercise Remark: In fact, $\forall A \in \mathcal{T}$, $\mathbb{P}^x(A) \in \{0,1\}$ and this prob. is ind. of $x \in \mathbb{R}^d$.

Reached here! However, this is not true for the germ field $\mathcal{F}^+(0)$.

Lecture 11 16.5.11 Scribe HW

Application of Markov prop. to local maxima thm. (p. 39).

Strong Markov prop. and reflection principle

Def. Stopping time in a filtered prob. space.

minimum of two stop. times Examples: Deterministic, increasing limit, discretization.

Which filtration to use? $\mathcal{F}^0(t)$ or $\mathcal{F}^+(t)$? $\mathcal{F}^0(t) \subseteq \mathcal{F}^+(t)$, hitting a closed set, hitting a open set.

Right continuity and alternative def. of stopping time.

Introduce $\mathcal{F}^+(T)$. Strong Markov Prop.

Skim proof? Reflection principle. The area of planar BM. Cor: planar BM does not visit points.

Zero set of BM is a perfect set.

Mart. in cont. time, Wald's lemmas for BM. Gambler's ruin for BM.

Remind Markov prop. Skorohod embedding - Azema-Yor embedding. Donsker's inv. principle.

Thm. (application of Markov prop. to local maxima): For a 1D BM, a.s.,

1) Every local max. is strict.
2) The set of times of local maxima is countable and dense.
3) The global max. is attained at a unique time.

proof: Everything follows from showing that if $I_1=[a,b]$, $I_2=[c,d]$ for given $a<b\leq c<d$ then a.s. $\max_{t\in I_1} B(t)\neq \max_{t\in I_2} B(t)$. To see this, note first that we may assume $b\neq c$. Indeed, since we showed $\mathbb{P}(\inf\{t>0 \mid B(t)>0\})=1$, (by the Markov prop.) Remove we see that $\max_{t\in I_2} B(t)=\max_{t\in[c+\frac{1}{n},d]} B(t)$ for some $n\in\mathbb{N}$. Hence we may consider each such $I_2'=[c+\frac{1}{n},d]$ separately. Assume $b\neq c$.

Denote $\hat{m}_1=\max_{t\in I_1} B(t)-B(b)$, $\hat{m}_2=\max_{t\in I_2} B(t)-B(c)$ and $X=B(c)-B(b)$.

By the Markov prop. at b and c, these are jointly ind. We have $\{\max_{t\in I_1} B(t)=\max_{t\in I_2} B(t)\}=\{X=\hat{m}_1-\hat{m}_2\}$. Since cond. on the RHS leaves X with a non-atomic dist., this event has prob. 0.

It follows that all pairs of disjoint intervals with rational endpoints have different maxima, (1) follows immediately.

To see (2), note that for a given $I=[a,b]$, $\max_{t\in I} B(t)\notin\{B(a),B(b)\}$.

This follows by the Markov prop. since we showed $\mathbb{P}(\inf\{t>0 \mid B(t)>0\}=0)=1$ and by considering the reversed BM. Hence every rational interval contains a local max., showing the denseness. Moreover, since loc. max. are strict, each one is the max. of some rational int., showing countability.

(3) follows since disjoint rational int. have diff. max. ∎

Stopping times and the strong Markov prop.

Def.: A RV T taking values in a filtered prob. space with filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is a stop. time wrt. $\{\mathcal{F}(t)\}$ if $\{T\leq t\}\in\mathcal{F}(t)$ $\forall t\geq 0$.

Examples: 1) Det. time. 2) Min. of stop. times.

3) Increasing limit: If T_n are stop. times and $T_n\uparrow T$ a.s. and $\mathcal{F}(t)$ is complete in the sense it contains all neg. sets then T is a stop. time since $\{T\leq t\}\underset{(\text{mod } 0)}{=}\bigcap_n\{T_n\leq t\}\in\mathcal{F}(t)$.

4) Upper approx.: For a stop. time T, let $T_n:=(m+1)2^{-n}$ if $m2^{-n}\leq T<(m+1)2^{-n}$. Then T_n is a stop. time $\forall n$.

Which σ-alg. to use? $\mathcal{F}^0(t)$ or $\mathcal{F}^+(t)$? Note $\mathcal{F}^0(t)\subseteq\mathcal{F}^+(t)$ $\forall t$.

First hit. time of a closed set is a stop. time wrt. $\mathcal{F}^0$, but inf. hit. time of an open set is a stop. time wrt. $\mathcal{F}^+$, but not $\mathcal{F}^0$.

We will use $\mathcal{F}^+(t)$ and also augment it by all neg. sets (but still call it $\mathcal{F}^+(t)$).

Note that $\mathcal{F}^+(t)$ is right-cont. in the sense that $\forall t$, $\mathcal{F}^+(t) = \bigcap_{\varepsilon>0} \mathcal{F}^+(t+\varepsilon)$.

Lemma: If T is a RV taking values in $[0,\infty]$ and satisfying $\{T<t\} \in \mathcal{F}(t)$ $\forall t$ wrt. a right-cont. filt. $\{\mathcal{F}(t)\}_{t\ge 0}$ then T is a stop. time wrt. $\{\mathcal{F}(t)\}$.

Proof: $\{T\le t\} = \bigcap_{k=1}^{\infty} \{T < t + \frac{1}{k}\} \in \bigcap_{k=1}^{\infty} \mathcal{F}(t+\frac{1}{k}) \overset{\text{right Cont.}}{=} \mathcal{F}(t)$.

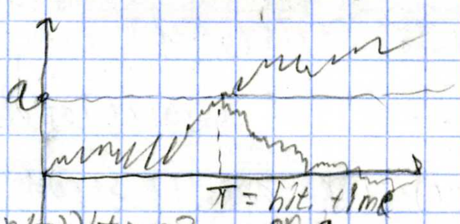
Def: For a stop. time T, $\mathcal{F}^+(T) = \{A \in \underset{\text{sigma-alg. of } \Omega}{\mathcal{A}} \mid A \cap \{T \le t\} \in \mathcal{F}^+(t)\ \forall t\}$.
All the events known by time T. Note, e.g., $(B(t) \mid t \le T) \in \mathcal{F}^+(T)$ and note we can write $\{T<t\}$ in the above def. by right-cont.

Thm (Strong Markov prop) For every a.s. finite stop. time the process $\{B(T+t) - B(T) \mid t \ge 0\}$ is a standard BM ind. of $\mathcal{F}^+(T)$.

Remark: There are "bad" times too, say $\tau = \max(0 \le t \le 1 \mid B(t) = 0)$.

Proof: Following BM book (P. 43).

Applications: 1) Reflection principle: If T is a stop. time and B a standard BM then B^*, called BM reflected at T and defined by $B^*(t) := B(t)\mathbb{1}_{(t\le T)} + (2B(T) - B(t))\mathbb{1}_{(t>T)}$ is also a standard BM.



Proof: For finite T, by the strong markov prop. both $\{B(t+T) - B(T) \mid t \ge 0\}$ and $\{-(B(t+T) - B(T)) \mid t \ge 0\}$ are standard BM ind. of $\{B(t) \mid 0 \le t \le T\}$.
Glueing the first to the end of $\{B(t) \mid 0 \le t \le T\}$ gives B.
Glueing the second gives B^*.

If T is not neccessarily finite, we can proceed by applying this to $T \wedge n$ and taking a limit $n \to \infty$. The limit is a.s. cont. and has the correct fin. dim. dist.

Cor.: If $a>0$ then $P(M(t)\ge a) = 2P(B(t) \ge a) = P(|B(t)| \ge a)$

Proof: apply refl. principle to first hitting time of a.

$$\{M(t)\ge a\} = \{B(t) \ge a\} \cup \underbrace{\{B(t) < a,\ M(t) \ge a\}}_{=\{B^*(t) > a\}}.$$

P. 148, Ex. I. 10 Add assump. that $\mathbb{E}X^2 < \infty$.

P. 45, Proof of thm. 2.21, calc. $P(M(t)\ge a)$ instead since it is only true mod 0 that $\{M(t) > a, B(t) \le a\} = \{B^*(t) \ge a\}$.

P. 46, Proof of thm. 2.24 seems it is not eq. in first display of proof and 4 needs to be 8. As in write up.
P. 47, B_2 not a BM by def. Replace $B(0)=0$ a.s. by a.e.

P. 127, thm. 5.15, natural filt. of BM, $\mathcal{F}^0$ or $\mathcal{F}^+$?

P. 57 In the optional stop. prop. 2.42, can T be infinite? Related to sol to ex 2.16.

(Lévy 1940)

2) Area of planar BM: The range of 2D BM has zero area, a.s.

Remark: But it has Hausdorff dim. 2. Much stronger, $\forall A \subseteq [0,\infty)$ closed $\dim B(A) = (2\dim A)\wedge d$. See McKean thm. 4.33 and Kaufman's thm. 9.28.

Def.: $\mathcal{L}_d$ = Leb. meas. on $\mathbb{R}^d$. $f*g$ = the conv. of f and g, $(f*g)(x)=\int f(y)g(x-y)dy$.

$A+x := \{a+x \mid a\in A\}$.

Lemma: If A_1, A_2 are Borel sets in $\mathbb{R}^d$ with pos. meas. then
$\mathcal{L}_d(\{x\in\mathbb{R}^2 \mid \mathcal{L}_d(A_1\cap(A_2+x))>0\})>0$.

Proof: Assume WLOG A_1, A_2 bdd. Note $\mathcal{L}_d(A_1\cap(A_2+x)) = 1_{A_1} * 1_{-A_2}(x)$.

By Fubini, $\int_{\mathbb{R}^d}(1_{A_1}*1_{-A_2})(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1_{A_1}(w)1_{-A_2}(x-w)\,dw\,dx =$
$= \int_{\mathbb{R}^d} 1_{A_1}(w)\int_{\mathbb{R}^d} 1_{-A_2}(x-w)\,dx\,dw = \mathcal{L}_d(A_1)\mathcal{L}_d(A_2)>0$

Proof of Lévy's thm: It is suff. to show that a.s. $\mathcal{L}_2(B[0,1])=0$.

Let $X=\mathcal{L}_2(B[0,1])$. First we check $\mathbb{E}X<\infty$. This follows from a tail estimate on the max. of a 1D BM W since

$$P(X>a)\le 2P\Big(\max_{t\in[0,1]}|W(t)|>\tfrac{\sqrt{a}}{2}\Big)\overset{\text{prev. cor.}}{\le} 8P\Big(W(1)>\tfrac{\sqrt{a}}{2}\Big)\le 8e^{-a/8}.$$

Second, by scale inv. $\{B(t)\}_t$ and $\{\sqrt{3}B(\frac{t}{3})\}_t$ have the same dist.

Hence $\mathbb{E}\mathcal{L}_2 B[0,3] = 3\mathbb{E}\mathcal{L}_2 B[0,1] = 3\mathbb{E}X$.

However, $\mathcal{L}_2 B[0,3]\le\sum_{j=0}^{2}\mathcal{L}_2 B[j,j+1]$ with equality iff $\mathcal{L}_2(B[i,i+1]\cap B[j,j+1])=0$ $\forall 0\le i<j\le 2$.

Moreover $3\mathbb{E}X=\mathbb{E}\mathcal{L}_2(B[0,3])\le\sum_{j=0}^{2}\mathbb{E}\mathcal{L}_2 B[j,j+1]=3\mathbb{E}X$.

Hence a.s. $\mathcal{L}_2(B[i,i+1]\cap B[j,j+1])=0$ $\forall 0\le i<j\le 2$.

Now consider $\{B(t)\mid t\in[0,1]\}$, $Y:=B(2)-B(1)$ and $\{B'(t)\mid t\in[0,1]\}$ defined by $B'(t):=B(t+2)-Y$. By the Markov prop., Y is ind. of both these processes since $B'(t)=[B(t+2)-B(2)]+B(1)$.

Let $R(x):=\mathcal{L}_2(B[0,1]\cap(x+B'[0,1]))$ and note $0=\mathbb{E}\mathcal{L}_2(B[0,1]\cap B[2,3])=\mathbb{E}R(Y)$.

Since Y is Gaussian, it follows $R(x)=0$ for a.e. x. By prev. lemma

We obtain that a.s. $\mathcal{L}_2 B[0,1]=0$ or $\mathcal{L}_2 B[2,3]=0$. Since these are ind., we are done.

Cor.: For any $d\ge 2$ and any $x,y\in\mathbb{R}^d$ $P_x(y\in B[0,1])=0$ (d-dim. BM does not hit points for $d\ge 2$)

Proof: By Proj, we may assume $d=2$. Note $\int_{\mathbb{R}^2}P_y(x\in B[0,1])dx \underset{\text{Fubini}}{=} \mathbb{E}_y\mathcal{L}_2 B[0,1] \underset{\text{Lévy}}{=} 0$

Hence for every y and a.e. x, $P_y(x\in B[0,1])=0$. Note

$P_y(x\in B[0,1])=P_0(x-y\in B[0,1])=P_0(y-x\in B[0,1])=P_x(y\in B[0,1])$.

Finally, to get every x, note $P_x(y\in B(0,1])=\lim_{\varepsilon\downarrow 0}P_x(y\in B[\varepsilon,1])=\lim_{\varepsilon\downarrow 0}\mathbb{E}_x P_{B(\varepsilon)}(y\in B[0,1-\varepsilon])=0$.

Reached here

3) zero set of 1D BM is a perfect set: Proof from BM book, p. 48, using strong Markov prop.

— Lecture 12 23.5.11 (Scribe) (HW)

Def. of martingale, submart. in cont. time.

Ex.: 1D BM is a martingale.

Prop.: (optional stop. thm) state but do not prove. Proof by approx., see BM book, p. 54.

Prop. (Wald's lemma).

Cor.: $S \le T$ stop. times with $\mathbb{E}T < \infty$:
$$\mathbb{E}(B(T)^2) = \mathbb{E}(B(S)^2) + \mathbb{E}(B(T) - B(S))^2$$

$B(t)^2 - t^2$ is a mart. for BM.

Second Wald's lemma.

Exit prob. and time for a 1D BM.

Skorohod embedding.

Def: A real-valued stochastic process $\{X(t) | t \ge 0\}$ is a martingale with respect to a filtration $\{\mathcal{F}(t) | t \ge 0\}$ if it is adapted to $\{\mathcal{F}(t)\}_{t \ge 0}$, $\mathbb{E}|X(t)| < \infty$ $\forall t \ge 0$ and for any $0 \le s \le t$, we have $\mathbb{E}(X(t) | \mathcal{F}(s)) = X(s)$ a.s.

Submartingale: $\ge$, supermartingale: $\le$

Example: 1D BM is a martingale wrt. $\mathcal{F}^+(t)$. For $s \le t$,
$$\mathbb{E}(B(t) | \mathcal{F}^+(s)) = \mathbb{E}(B(t) - B(s) | \mathcal{F}^+(s)) + B(s) \overset{\text{Markov prop. wrt. } \mathcal{F}^+(s)}{=} \mathbb{E}(B(t) - B(s)) + B(s) = B(s).$$

Def.: The martingale $\{X(t)\}_{t \ge 0}$ is called cont. if a.s. its sample paths are cont..

Thm. (optional stopping): If $\{X(t)\}_{t \ge 0}$ is a cont. mart. and T is a stopping time s.t. there exists an int. X satisfying $|X(t \wedge T)| \le X$ a.s for every t, then $\mathbb{E}X(T) = X(0)$.

exercise

Proof: Use approx. by a discrete-time mart. and corresponding result in discrete time. Left as exercise.

Remark: It is also true that if we have two stopping times $S \le T$ a.s. and $|X(t \wedge T)| \le X$ for an int. X

then $\mathbb{E}(X(T)|\mathcal{F}(S)) = X(S)$ a.s. This can again be proved using the corresp. discrete-time result.

Prop. (Wald's lemma for BM): If T is a stopping time for 1D BM (wrt. $\mathcal{F}^+$) and $\mathbb{E}T<\infty$ then $\mathbb{E}B(T)=0$.

Remark: In fact, this is still true under the assump. $\mathbb{E}\sqrt{T}<\infty$.

Note it is not true for hitting time T_1 of 1 though $\mathbb{E}T^{\frac{1}{2}-\varepsilon}<\infty$ $\forall\, 0<\varepsilon<\frac{1}{2}$.

Proof: Let $M_k := \max\limits_{0\le t\le 1} |B(t+k)-B(k)|$ and $M=\sum\limits_{k=1}^{\lceil T\rceil} M_k$.

Note $\forall t$, $|B(T\wedge t)|\le M$ a.s. so if $\mathbb{E}M<\infty$ the Prop. will follow from the optional stop. thm. We have

$$\mathbb{E}M=\sum_{k=1}^{\infty}\mathbb{E}M_k 1_{(T>k-1)} \underset{\text{markov prop. and } \{T>k-1\}\in\mathcal{F}^+(k-1)}{=} \sum_{k=1}^{\infty}\mathbb{E}M_k\mathbb{P}(T>k-1)=\mathbb{E}M_0\cdot\mathbb{E}(T+1)$$

and $\mathbb{E}M_0<\infty$ since $\mathbb{P}(M_0>a)=\mathbb{P}(|B(1)|>a)$.

Lemma: For a 1D BM, $\{B(t)^2-t\}_{t\ge 0}$ is a mart. (wrt. $\mathcal{F}^+$).

Proof: Adaptedness and int. are clear. For all $0\le s\le t$:

$$\mathbb{E}(B(t)^2-t|\mathcal{F}^+(s))=\mathbb{E}((B(t)-B(s))^2|\mathcal{F}^+(s))+\underbrace{2\mathbb{E}(B(s)B(t)|\mathcal{F}^+(s))}_{=0}-B(s)^2-t$$

$$\overset{\text{Markov prop.}}{=} t-s-B(s)^2-t=B(s)^2-s.$$

Prop. (Wald's second lemma for BM): If T is a stop. time for 1D BM (wrt. $\mathcal{F}^+$) and $\mathbb{E}T<\infty$ then $\mathbb{E}B(T)^2=\mathbb{E}T$.

Proof: Define the stop. times $H_n := \min\{t\ge 0 \mid |B(t)|=n\}$ and $T_n := T\wedge H_n$.

Note $|B(t\wedge T_n)^2 - t\wedge T_n|\le n^2+\overset{\leftarrow\text{ integrable}}{T}$ a.s. Hence by optional stop.

$\mathbb{E}B(T_n)^2=\mathbb{E}T_n$. By Fatou's lemma $\mathbb{E}B(T)^2\le \liminf\limits_{n\to\infty}\mathbb{E}B(T_n)^2=\liminf\limits_{n\to\infty}\mathbb{E}T_n\le\mathbb{E}T$.

For the other dir, note first that

$$\mathbb{E}B(T)^2=\mathbb{E}(B(T)-B(T_n))^2+\mathbb{E}B(T_n)^2+2\mathbb{E}B(T_n)(B(T)-B(T_n))\ge\mathbb{E}B(T_n)^2$$

Since $\mathbb{E}B(T_n)(B(T)-B(T_n))=\mathbb{E}B(T_n)\mathbb{E}(B(T)-B(T_n)|\mathcal{F}^+(T_n))=0$

by the strong Markov prop. and Wald's lemma ($\mathbb{E}(T-T_n|\mathcal{F}^+(T_n))<\infty$ a.s.)

Hence $\mathbb{E}B(T)^2\ge\lim\limits_{n\to\infty}\mathbb{E}B(T_n)^2=\lim\limits_{n\to\infty}\mathbb{E}T_n \overset{\text{Mono. Conv.}}{=}\mathbb{E}T$.

Cor. (Hitting prob. and time): For $a,b>0$, let $T=\min\{t\ge 0 \mid B(t)\in\{-a,b\}\}$. Then

1) $\mathbb{P}(B(T)=a)=\frac{b}{a+b}$, $\mathbb{P}(B(T)=b)=\frac{a}{a+b}$.

2) $\mathbb{E}T=ab$.

Proof: Same as for SRW using Wald's lemmas.

Skorohod embedding

Moving towards the Donsker inv. principle, we seek first to solve the Skorohod embed. problem. Given a RV X, we seek an integrable stopping time T for BM s.t. $B(T) \sim X$.

By Wald's lemmas, $\mathbb{E}B(T)=0$ and $\mathbb{E}B(T)^2=\mathbb{E}T$, hence this can only be possible for centered X with finite var.

Thm.: If X satisfies $\mathbb{E}X=0, \mathbb{E}X^2<\infty$ then there exists an int. stop. time T for 1D BM wrt. $\mathcal{F}^+$ s.t. $B(T)\sim X$ (and $\mathbb{E}T = \mathbb{E}X^2$).

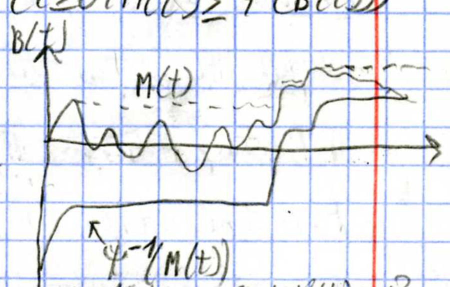
Example: If X takes only two values $-a, b$ then $\min(t\geq 0 \mid B(t)\in\{-a,b\})$ has the required prop. (in particular for $a=b=1$).

There are two well-known proofs of the prev. thm.: the Dubins embedding thm. and the Azema-Yor embed. thm.. We will present the Azema-Yor embed. See Durrett or Mörters-Peres for the other approach.

Thm. (Azema-Yor embed.): If X satisfies $\mathbb{E}X=0, \mathbb{E}X^2<\infty$, let

$$\psi(x) = \begin{cases} \mathbb{E}(X \mid X\geq x) & \text{if } \mathbb{P}(X\geq x)>0 \\ 0 & \text{o/w} \end{cases}$$

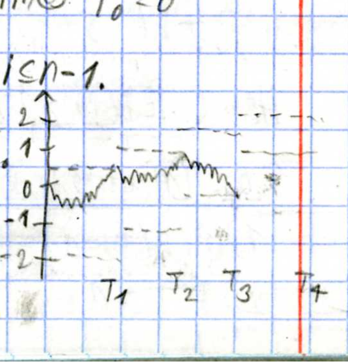
For a 1D BM B, let M be its max. proc. $M(t)=\max_{0\leq s\leq t} B(s)$ and define a stop. time T by $T := \inf(t\geq 0 \mid M(t)\geq \psi(B(t)))$ then $\mathbb{E}T=\mathbb{E}X^2$ and $B(T)\sim X$.



$\psi^{-1}(b) = \sup\{b \mid \psi(b)\leq x\}$.

Proof: We start by proving the thm. when X is fin. supp.

Lemma: Suppose X takes only the values $x_1<x_2<\dots<x_n$ and $\mathbb{E}X=0$. Define $y_1<\dots<y_{n-1}$ by $y_i=\psi(x_{i+1})$ and define stop. times $T_0=0$ and $T_i=\inf\{t\geq T_{i-1} \mid B(t)\notin (x_i, y_i)\}$ for $i\leq n-1$. Then T_{n-1} satisfies $\mathbb{E}T_{n-1}=\mathbb{E}X^2$ and $B(T_{n-1})\sim X$.



Uniform dist. on $\{-2,-1,0,1,2\}$

Here $T=T_3=T_4$ and $B(T)=0$.

Proof: First note $y_i \geq x_{i+1}$ with equality iff $i=n-1$.

We have $\mathbb{E}\, T_{n-1}<\infty$ and hence $\mathbb{E}\, B(T_{n-1})^2=\mathbb{E}\, T_{n-1}$. For $i=1,\dots,n-1$

let $Y_i=\begin{cases}\mathbb{E}(X \mid X\geq x_{i+1}) & \text{if } X\geq x_{i+1}\\ X & \text{if } X\leq x_i\end{cases}$

Note Y_1 has $\mathbb{E}\, Y_1=0$ and $Y_1\in\{x_1,y_1\}$. Similarly, for $i\geq 2$, given $Y_{i-1}=y_i$ we have $\mathbb{E}\, Y_i=y_{i-1}$ and $Y_i\in\{x_i,y_i\}$ and given $Y_{i-1}=x_j$ for some $j\leq i-1$ we have $Y_i=x_j$. Finally note $Y_{n-1}=X$.

Hence the lemma will follow by showing $(B(T_1),\dots,B(T_{n-1}))\overset{d}{=}(Y_1,\dots,Y_{n-1})$.

This is immediate from the above discussion (since a RV taking two given values is determined by its expectation).

Lemma: T_{n-1} from the prev. lemma equals T of the Azéma-Yor thm.

Proof: Let j be such that $B(T_{n-1})=x_j$. Note $\Psi(B(T_{n-1}))=y_{j-1}$.

If $j\leq n-1$ then $T_{j-1}<T_j=\dots=T_{n-1}$ and $B(T_{j-1})=y_{j-1}$.

If $j=n$ then $B(T_{n-1})=x_n=y_{n-1}$. Hence in both cases $M(T_{n-1})\geq y_{j-1}$.

It follows that $T\leq T_{n-1}$ since $y_{j-1}=\Psi(x_j)=\Psi(B(T_{n-1}))$.

Conversely if $T_{i-1}\leq t\leq T_i$ for some $i\leq j$ then $B(t)\in(x_i,y_i)$ and we deduce $M(t)<y_i\leq\Psi(B(t))$. Hence $T\geq T_{n-1}$ establishing the lemma.

Exercise

The rest of the proof is to use a limiting process to pass to general RVs. This is left as an exercise.

The Donsker invariance principle

Let $(X_n)_{n\geq 1}$ be a sequence of IID RVs with $\mathbb{E}\, X_1=0$ and $\mathrm{Var}(X_1)=1$. Denote $S_n=\sum_{i=1}^n X_i$, $S(t)$ as its linear interpolation $S(t):=S_{\lfloor t\rfloor}+(t-\lfloor t\rfloor)(S_{\lfloor t\rfloor+1}-S_{\lfloor t\rfloor})$. Note $S\in C[0,\infty)$. Now define $\{S_n^*\mid n\geq 1\}$ of random fcns. in $C[0,1]$ by $S_n^*(t):=\frac{S(nt)}{\sqrt{n}}$, $t\in[0,1]$.

Functional central limit thm. or inv. principle

Thm: S_n^* conv. in dist. to a std. BM B on $[0,1]$ in the space $C[0,1]$ (with sup norm).

Remark: This means that for any bdd. cont. function g on $C[0,1]$ we have $\mathbb{E}\, g(S_n^*)\xrightarrow[n\to\infty]{}\mathbb{E}\, g(B|_{[0,1]})$. Moreover, this holds also for general bdd. meas. g with $\mathbb{P}(g \text{ is discont at } B)=0$.

Portmanteau thm.

It is also equivalent to having $\forall$ closed $K\subseteq C[0,1]$, $\limsup_{n\to\infty}\mathbb{P}(S_n^*\in K)\leq\mathbb{P}(B|_{[0,1]}\in K)$

Exercise: Embedding lemma 5.24. Proof of Donsker's inv. principle.
Dist. of max. of walk up to n. LIL in HW.

Lecture 13 6.6.11 scribe HW sol.? Teaching survey

Stochastic integration

Consider SRW on $\mathbb{Z}$, S_n. What is dS_n? It is the random increments ± 1 unif. ind.

Consider a gambling strategy $R_n = \sum_{i=1}^{n} b_i X_i$. Can write $R_n = \sum_1^n b_i dS_i$

Here $\{b_i\}_{i=1}^n$ is a random predictable seq., $b_i \in \sigma(X_1, \dots, X_{i-1})$, X_i ± 1 unif. ind.

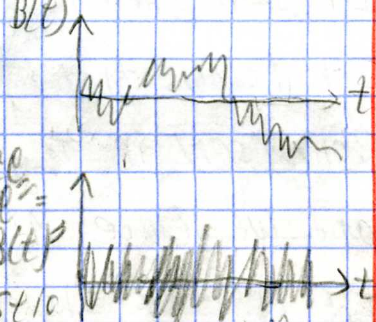
Thus R_n is the money gained by time n and $\{R_i\}_{i=1}^n$ is a martingale.

We would similarly want to define $\int_0^t H(s)dB(s)$.

However, B is nowhere diff., hence of unbdd. variation.
Cannot define a Stieltjes integral.

We will see that for suitable integrands we will be able to define the integral as a random process.

Denote by $(\Omega, \mathcal{A}, \mathbb{P})$ our prob. space.



Suppose $\{\mathcal{F}(t)\}_{t\geq 0}$ is a filtration to which our BM is adapted, such that the SMP holds, and which is complete

Stochastic integral makes this a form of dist.

($\forall A \in \mathcal{A}$, $\mathbb{P}(A)=0$ we have $A \in \mathcal{F}(t)$ $\forall t$).

E.g., $\mathcal{F}$ = completion of $\mathcal{F}^+$.

Def. $\{X(t,\omega) \mid t\geq 0, \omega\in\Omega\}$ is progressively measurable if $\forall t \geq 0$, the mapping $X: [0,t]\times\Omega \to \mathbb{R}$ is measurable wrt. $\mathcal{B}([0,t])\otimes\mathcal{F}(t)$. Borel sets

Remark: Such X are in particular adapted to $\mathcal{F}(t)$. ($X(t)\in\mathcal{F}(t)$ $\forall t$)

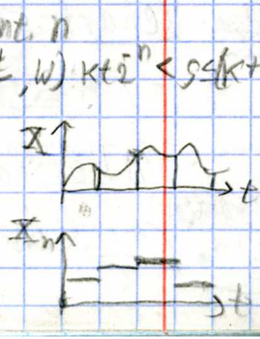
Lemma: Any X which is adapted and either left or right cont. is prog. meas.

Proof: Assume WLOG X is right-cont. Fix $t>0$. For a positive int. n and $0\leq s\leq t$, let $X_n(0,\omega) = X(0,\omega)$ and $X_n(s,\omega) := X\left(\frac{(k+1)t}{2^n},\omega\right)$ $kt2^{-n} < s \leq (k+1)t2^{-n}$.

$(s,\omega)\mapsto X_n(s,\omega)$ is $\mathcal{B}([0,t])\times\mathcal{F}(t)$ meas. by adaptedness.

By right-cont., $\lim_{n\uparrow\infty} X_n(s,\omega) = X(s,\omega)$ $\forall s\in[0,t]$, $\omega\in\Omega$.

Hence $(s,\omega)\mapsto X(s,\omega)$ is $\mathcal{B}([0,t])\times\mathcal{F}(t)$ meas.



We will define $\int_0^\infty H(s)\,dB(s)$ for H prog. meas. s.t. $\|H\|_2^2 := \mathbb{E}\int_0^\infty H(s)^2\,ds < \infty$.
Then $\int_0^t H(s)\,dB(s) := \int_0^\infty H(s)\mathbb{1}_{[0,t]}(s)\,dB(s)$. This is far from the most general def. possible and other int. are useful too! However, it has the advantages that it is an isometry in L_2: $\mathbb{E}\left[\left(\int_0^\infty H(s)\,dB(s)\right)^2\right] = \|H\|_2^2$, and that $\{\int_0^t H(s)\,dB(s) \mid t \geq 0\}$ is a cont. mart. wrt. $\mathcal{F}(t)$. Finally, it is easier to define and already useful.

Itô integral. Perhaps mention also Stratonovich.

We begin by defining for $H(t,\omega) := \sum_{i=1}^{k} A_i(\omega)\mathbb{1}_{(t_i, t_{i+1}]}(t)$ $0 \leq t_1 \leq \ldots \leq t_{k+1}$ and $A_i \in \mathcal{F}(t_i)$ (predictable)

step proc. prog. meas.

$$\int_0^\infty H(s)\,dB(s) := \sum_{i=1}^{k} A_i(B(t_{i+1}) - B(t_i))$$

Now, we will show that a prog. meas. H with $\|H\|_2^2 < \infty$ can be approx. in L_2 by prog. meas. step proc. H_n. We then define $\int_0^\infty H(s)\,dB(s) = L_2\text{-}\lim_n \int_0^\infty H_n(s)\,dB(s)$ and show the limit exists and is ind. of the approx. seq.

<u>Lemma:</u> For a prog. meas. H with $\|H\|_2^2 < \infty$ there exists a seq. $\{H_n\}_{n\geq 0}$ of prog. meas. step proc. s.t. $\|H - H_n\|_2 \xrightarrow[n\to\infty]{} 0$.

<u>Proof:</u> We approx. H by 1) a bdd. prog. meas. proc.
2) a bdd. a.s. cont. prog. meas. proc.
3) a prog. meas. step process.

p. 192, l. 9, truncation only from above

1) For $n > 0$, let $H_n(s,\omega) := H(s,\omega)\mathbb{1}_{[0,n]}(s)$. Then $H_n \xrightarrow[n\to\infty]{L_2} H$.
For $m > 0$, we let $H_{n,m}(s,\omega) := \begin{cases} H(s,\omega) & |H(s,\omega)| \leq m \\ 0 & \text{o/w} \end{cases}$.
Then $H_{n,m} \xrightarrow[m\to\infty]{L_2} H_n$.

2) Assume H is unif. bdd. and on a compact int. Let $h = \frac{1}{n}$ and, denoting $H(s,\omega) := H(0,\omega)$ for $s < 0$, let $H_n(s,\omega) := \frac{1}{h}\int_{s-h}^{s} H(t,\omega)\,dt$.
H_n is still prog. meas. (int. over past) and is a.s. cont.
Since $\lim_{h\downarrow 0} H_n(s,\omega) = H(s,\omega)$ for a.e. s, a.s., we obtain from the bdd. conv. thm. that $H_n \xrightarrow[n\to\infty]{L_2} H$.

3) Assume H is unif. bdd., a.s. cont. and on a compact int.
Let $H_n(s,\omega) := H(i/n, \omega)$ for $i/n \leq s < \frac{i+1}{n}$. H_n is prog. meas. and $H_n \xrightarrow[n\to\infty]{L_2} H$ again by bdd. conv.

<u>Lemma:</u> For a prog. meas. step process H with $\|H\|_2^2 < \infty$ we have $\mathbb{E}\left[\int_0^\infty H(s)\,dB(s)\right]^2 = \|H\|_2^2$.

<u>Proof:</u> Write $H = \sum_{i=1}^{k} A_i \mathbb{1}_{(a_i, a_{i+1}]}$. $\mathbb{E}\left[\int_0^\infty H(s)\,dB(s)\right]^2 = \mathbb{E}\sum_{i,j=1}^{k} A_i A_j (B(a_{i+1}) - B(a_i))(B(a_{j+1}) - B(a_j)) =$

$$= \underbrace{2\sum_{i=1}^{k}\sum_{j=i+1}^{k} \mathbb{E}A_i A_j (B(a_{i+1}) - B(a_i))(B(a_{j+1}) - B(a_j))}_{=0 \text{ by the Markov prop. and pred.}} + \sum_{i=1}^{k}\mathbb{E}[A_i^2(B(a_{i+1}) - B(a_i))^2] \underset{\text{predictability}}{=} \sum_{i=1}^{k}\mathbb{E}A_i^2(a_{i+1} - a_i) = \|H\|_2^2.$$

Cor.: If $\{H_n\}_{n\geq 0}$ are prog. meas. step fcns. with $\|H_n - H_m\|_2 \xrightarrow[n,m\to\infty]{} 0$ then $\mathbb{E}[(\int_0^\infty (H_n(s) - H_m(s))dB(s))^2] \xrightarrow[n,m\to\infty]{} 0$.

Thm.: If $H, \{H_n\}_{n\geq 0}$ are prog. meas., H_n step proc., $H_n \xrightarrow[n\to\infty]{L_2} H$ then $\lim_{n\to\infty} \int_0^\infty H_n(s)dB(s) =: \int_0^\infty H(s)dB(s)$ exists in L_2, is ind. of the choice of $\{H_n\}_{n\geq 0}$ and $\mathbb{E}(\int_0^\infty H(s)dB(s))^2 = \|H\|_2^2$.

Proof: Immediate conseq. of Lemma, cor. and the comp. of L_2.

Remark: If $H_n \xrightarrow{L_2} H$ suff. fast, that is, $\sum_{n=1}^\infty \mathbb{E}\int_0^\infty (H_n(s) - H(s))^2 ds < \infty$ then $\sum_{n=1}^\infty \mathbb{E}(\int_0^\infty (H_n(s) - H(s))dB(s))^2 < \infty$ and hence a.s., $\sum_{n=1}^\infty (\int_0^\infty H_n dB - \int_0^\infty H dB)^2 < \infty$ and hence $\int_0^\infty H_n(s)dB(s) \xrightarrow[n\to\infty]{} \int_0^\infty H(s)dB(s)$.

Def: For a prog. meas. H with $\mathbb{E}\int_0^t H(s,\omega)^2 ds < \infty$ we let $\int_0^t H(s)dB(s) := \int_0^\infty H(s)\mathbb{1}_{\{s\leq t\}}dB(s)$

Def: We say that a stoch. proc. X is a modification of the stoch. proc. Y if $P(X(t) = Y(t)) = 1 \ \forall t$.

Thm: If H is prog. meas. and $\mathbb{E}\int_0^t H(s,\omega)^2 ds < \infty \ \forall t\geq 0$ then there is an a.s. cont. modification of $\{\int_0^t H(s)dB(s)\}_{t\geq 0}$. Moreover this process is a mart. and hence, in part., $\mathbb{E}\int_0^t H(s)dB(s) = 0 \ \forall t\geq 0$.

We will need Doob's L_p maximal inequality: For a cont. mart. X and $p>1$ then for any $t\geq 0$, $\mathbb{E}(\sup_{0\leq s\leq t} |X(s)|^p) \leq (\frac{p}{p-1})^p \mathbb{E}|X(t)|^p$.

Proof by the discrete version and approx.

Proof: Fix some t_0 and let $\{H_n\}_{n\geq 0}$ be step proc. s.t. $\|(H_n - H)\mathbb{1}_{\{\cdot\leq t_0\}}\|_2 \to 0$

Then $\mathbb{E}(\int_0^{t_0} (H_n(s) - H(s))dB(s))^2 \xrightarrow[n\to\infty]{} 0$ (*).

Since $\forall s\leq t$, $\int_0^s H_n(u)dB(u)$ is $\mathcal{F}(s)$ meas. and $\int_s^t H_n(u)dB(u)$ is ind. of $\mathcal{F}(s)$, then $\{\int_0^t H_n(u)dB(u)\}_{0\leq t\leq t_0}$ is a martingale, $\forall n$.

Define $X(t) := \mathbb{E}(\int_0^{t_0} H(s)dB(s) \mid \mathcal{F}(t))$ for $0\leq t\leq t_0$. X is also a mart. and $X(t_0) = \int_0^{t_0} H(s)dB(s)$. By the L_2 max. ineq.

$\mathbb{E}(\sup_{0\leq t\leq t_0} (\int_0^t H_n(s)dB(s) - X(t))^2) \leq 4\mathbb{E}[(\int_0^{t_0} (H_n(s) - H(s))dB(s))^2] \xrightarrow{\text{by } (*)} 0$.

By taking a subseq. we see that X is the a.s. $\underset{\text{unif.}}{\text{limit}}$ of cont. process and hence is a.s. cont.. By taking L_2 limits $\int_0^t H(s)dB(s)$ is $\mathcal{F}(t)$ meas. and $\int_t^{t_0} H(s)dB(s)$ is ind. of $\mathcal{F}(t)$ with zero expect.

Hence $X(t) = \int_0^t H(s)dB(s)$.

Itô's Formula

Fundamental thm. of calc.: $f(x(t))-f(x(0))=\int_0^t f'(x(s))\,dx(s)$

when x is cont. and of bdd. variation.

Thm. (Itô's Formula I): For a C_2 $f:\mathbb{R}\to\mathbb{R}$ with $\mathbb{E}\int_0^t f'(B(s))^2\,ds<\infty$ for some $t>0$

we have a.s. for all $0\le s\le t$:

$$f(B(s))-f(B(0))=\int_0^s f'(B(u))\,dB(u)+\frac{1}{2}\int_0^s f''(B(u))\,du.$$

Thm. (Itô's Formula II): Suppose $\{g(s)\mid s\ge 0\}$ is an a.s. increasing cont. proc.

Let $f:\mathbb{R}^2\to\mathbb{R}$ be $C_{2,1}$. Assume $\mathbb{E}\int_0^t[\partial_x f(B(s),g(s))]^2\,ds<\infty$

for some $t>0$. Then a.s. for all $0\le s\le t$,

$$f(B(s),g(s))-f(B(0),g(0))=\int_0^s \partial_x f(B(u),g(u))\,dB(u)+\int_0^s \partial_y f(B(u),g(u))\,dg(u)+$$
$$+\frac{1}{2}\int_0^s \partial_{xx} f(B(u),g(u))\,du.$$

Multidimensional Itô's formula: $f:\mathbb{R}^{d+m}\to\mathbb{R}$ $C_{\underbrace{2,2,\dots,2}_{d},\underbrace{1,1,\dots,1}_{m}}$. Denote

$\nabla_x f=(\partial_1 f,\dots,\partial_d f)$, $\nabla_y f=(\partial_{d+1}f,\dots,\partial_{d+m}f)$,

$$\int_0^t \nabla_x f(B(u),g(u))\cdot dB(u):=\sum_{i=1}^d\int_0^t \partial_i f(B(u),g(u))\,dB_i(u)$$

$$\int_0^t \nabla_y f(B(u),g(u))\cdot dg(u)=\sum_{i=1}^m\int_0^t \partial_{d+i} f(B(u),g(u))\,dg_i(u)$$

and $\Delta_x f=\sum_{j=1}^d \partial_{jj}f$. Then if B is a d-dim. BM, $\{g(s)\}_{s\ge 0}$ is cont.

adapted with values in $\mathbb{R}^m$ and increasing comp., and for some $t>0$

$\mathbb{E}\int_0^t|\nabla_x f(B(u),g(u))|^2\,ds<\infty$ then a.s. $\forall\, 0\le s\le t$:

$$f(B(s),g(s))-f(B(0),g(0))=\int_0^s \nabla_x f(B(u),g(u))\cdot dB(u)+$$
$$+\int_0^s \nabla_y f(B(u),g(u))\cdot dg(u)+\frac{1}{2}\int_0^s \Delta_x f(B(u),g(u))\,du.$$

Remark: Since the formula holds a.s. $\forall s\in[0,t]$, it holds also at stop. times bdd. by t. In part., if $f:U\to\mathbb{R}$ satisfies the diff. cond. on an open set U and $K\subset U$ is compact then we may replace f by $f^*:=f\cdot g$, where g is smooth with compact supp. in U, $g\equiv 1$ on K. Let T be the first exit time from K. It follows that the thm. applies to f^*, and hence to f, at times $s\wedge T$ for all $s<\infty$ simultaneously.